Crash course: the hyperfinite II_1 factor and the standard form

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Abstract

(Masterclass "Topological quantum field theories, quantum groups and 3-manifold invariants") This is a follow-up notes for the introductory talks (45min × 2) on the hyperfinite II_1 factor and the standard form prepared for Prof. Ryszard Nest's talks Oct.9-10. There may be typos or errors.

1 Talk 1 (Oct. 6 2014, 10:30-11:15)

We define the notion of II_1 factors and see one example, namely the hyperfinite II_1 factor. We use the symbol H for complex Hilbert space (all Hilbert spaces in this notes are separable, and infinite-dimensional). The inner product $\langle\ \cdot\ ,\ \cdot\ \rangle\colon H\times H\to\mathbb{C}$ is assumed to be linear in the first variable and conjugate-linear in the second variable. We use Greek letters $\xi,\eta,\cdots$ for vectors in H and Italic letters $x,y,\cdots$ for bounded linear operators on H.

A linear opeartor $x\colon H\to H$ is *bounded*, if it is continuous in Hilbert space norm. In this case, there exists $c>0$ such that $\|x\xi\|\le c\|\xi\|$ $(\xi\in H)$. The infimum of such c is called the *(operator) norm* of x, denoted as $\|x\|$:

$$\|x\|=\sup_{\xi\in H,\ \|\xi\|\le 1}\|x\xi\|.$$

Definition 1.1. We denote by $\mathbb{B}(H)$ the set of all bounded linear maps from H to itself.

$\mathbb{B}(H)$ equipped with the operator norm is a Banach space. For each $x\in\mathbb{B}(H)$ there exists a unique $x^*\in\mathbb{B}(H)$ satisfying

$$\langle x\xi,\eta\rangle=\langle\xi,x^*\eta\rangle\ (\xi,\eta\in H).$$

x^* is called the *adjoint* of x. The map $x\mapsto x^*$ has the following properties $(x,y\in\mathbb{B}(H),\ \lambda,\mu\in\mathbb{C})$:

(1) $(\lambda x+\mu y)^*=\overline{\lambda}x^*+\overline{\mu}y^*$.

(2) $(x^*)^*=x$.

(3) $(xy)^*=y^*x^*$.

(4) $\|x^*x\|=\|x\|^2$.

$\mathbb{B}(H)$ has the *-algebra structure with respect to the operator sum, multiplication, etc. A vector subspace A of $\mathbb{B}(H)$ is called a **-subalgebra* if A is closed under operator product and *. For simplicity we only consider *-subalgebras of $\mathbb{B}(H)$ containing the unit $1=1_H$.

Definition 1.2. A *-subalgebra A of $\mathbb{B}(H)$ with unit 1_H is called a C^**-algebra*, if A is closed in operator norm topology. A is called a *von Neumann algbera* on H, if A is closed with respect to the strong operator topology (SOT).
Here, a net $\{x_i\}_{i\in I}\subset\mathbb{B}(H)$ converges to $x\in\mathbb{B}(H)$ with respect to SOT, if $\lim_i\|x_i\xi-x\xi\|=0$ for every $\xi\in H$.

Below we always assume that all C^*-algebras on H contain the unit $1=1_H$. Next lemma is only relevant for Corollary 1.5, which is not important either for understanding the rest of the topics. So they can safely be ignored.

Definition 1.3. Let $\mathcal{S} \subset \mathbb{B}(H)$. The *commutant* $\mathcal{S}'$ of $\mathcal{S}$ is the set $\mathcal{S}' := \{a \in \mathbb{B}(H); ax = xa, \quad \forall x \in \mathcal{S}\}$.

If a subset $\mathcal{S} \subset \mathbb{B}(H)$ is self-adjoint (i.e. $x \in \mathcal{S} \Rightarrow x^* \in \mathcal{S}$), then $\mathcal{S}'$ is a von Neumann algebra. The next theorem shows that von Neumann algbera is characterized by its double commutant:

Theorem 1.4 (von Neumann's double commutant theorem)**.** *Let M be a *-subagebra of $\mathbb{B}(H)$ containing unit 1_H. Then $\overline{M}^{\rm SOT} = M''(= (M')')$.*

Proof. $\overline{M}^{\rm SOT} \subset M''$: Let $x \in \overline{M}^{\rm SOT}$. Then there exists a net $(x_i)_{i\in I} \subset M$ converging to x strongly. Then for every $y' \in M'$ and $\xi \in H$, we have

$$xy'\xi = \lim_{i\to\infty} x_i y'\xi = \lim_{i\to\infty} y' x_i \xi = y'x\xi.$$

Since ξ is arbitrary, $xy' = y'x$ holds. Therefore $x \in (M')'$.
$\overline{M}^{\rm SOT} \supset M''$: Let $x \in M''$. We show that x is in the strong closure of M. This, by definition of SOT, amounts to proving the following: For every $\varepsilon > 0, n \in \mathbb{N}$ and $\xi_1, \cdots, \xi_n \in H$, there exists $x_0 \in M$ such that

$$\|x\xi_i - x_0\xi_i\| < \varepsilon, \qquad 1 \le i \le n. \tag{1}$$

Fix $\varepsilon > 0$ and $\xi_1, \cdots, \xi_n \in H$. Let $H_n = H \oplus H \oplus \cdots \oplus H$ (n copies) and define $\pi\colon \mathbb{B}(H) \to \mathbb{B}(H_n)$ by

$$\pi(a)(\eta_1, \cdots, \eta_n) = (a\eta_1, \cdots, a\eta_n) \ (a \in \mathbb{B}(H)), \eta_1, \cdots, \eta_n \in H.$$

That is, $\pi(a) = \mathrm{diag}(a, a, \cdots, a)$ regarded as an $n \times n$ matrices with entries in $\mathbb{B}(H)$. It is straightforward to check that the commutant of $\pi(M)$ is the set of all $n \times n$ matrices with entries in M':

$$\pi(M)' = \left\{[x'_{ij}]; x'_{ij} \in M', 1 \le i, j \le n\right\} \subset \mathbb{B}(H_n) \cong M_n(\mathbb{B}(H)).$$

Let $E := \overline{\pi(M)(\xi_1, \cdots, \xi_n)} \subset H_n$, and let e be the projection of H_n onto E. Then E is $\pi(M)$-invariant. Therefore $E^\perp$ is also $\pi(M)$-invariant. This shows that $e \in \pi(M)'$. Thus $e = [e_{ij}]$, where $e_{ij} \in M'$. It is then clear that since $x \in M''$, $\pi(x) = \mathrm{diag}(x, x, \cdots, x)$ commutes with e:

$$\pi(x)e = [xe_{ij}] = [e_{ij}x] = e\pi(x).$$

It then follows that (since $1 \in M$, $(\xi_1, \cdots, \xi_n) \in E$ holds)

$$\pi(x)(\xi_1, \cdots, \xi_n) = \pi(x)e(\xi_1, \cdots, \xi_n) = e\pi(x)(\xi_1, \cdots, \xi_n) \in E.$$

Therefore there exists $x_0 \in M$ such that

$$\|(\pi(x) - \pi(x_0))(\xi_1, \cdots, \xi_n)\|^2 = \sum_{i=1}^n \|x\xi_i - x_0\xi_i\|^2 < \varepsilon^2.$$

This implies (1). Therefore $x \in \overline{M}^{\rm SOT}$. □

Corollary 1.5. *Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra, and let $x = x^* \in M$ be its self-adjoint element. Let $x = \int_{\mathrm{sp}(x)} \lambda de(\lambda)$ be the spectral decomposition of x. Then $e(J) \in M$ for every Borel subset $J \subset \mathbb{R}$.*

Proof. Denote by $\mathrm{Proj}(H)$ the set of all orthogonal projections on H. Let $\mathcal{B}(\mathbb{R})$ be the σ-algebra of all Borel subsets of $\mathbb{R}$. Fix $u \in \mathcal{U}(M')$ ($\mathcal{U}(M')$ is the group of unitary elements in M'). Define a new map $\tilde{e}\colon \mathcal{B}(\mathbb{R}) \to \mathrm{Proj}(H)$ by

$$\tilde{e}(J) = ue(J)u^*, \qquad J \in \mathcal{B}(\mathbb{R}).$$

It is straightforward to see that $\tilde{e}(\cdot)$ is a projection valued measure, and $\tilde{e}(\cdot)$ is the spectral resolution of $\tilde{x} := uxu^*$. Since $x \in M, u \in M'$, we have

$$uxu^* = \int_{\mathrm{sp}(x)} \lambda d\tilde{e}(\lambda) = x.$$

This shows, by the uniqueness of spetral decomposition, that $\tilde{e} = e$. That is, $ue(J)u^* = e(J)$ for every $u \in \mathcal{U}(M')$. It is also not difficult to show that M' is spanned by $\mathcal{U}(M')$ (cf. proof sketched in Lemma 1.12), the unitary elements of M'. Therefore $e(J) \in (M')' = M$ by double commutant theorem. □

Definition 1.6. For a self-adjoint subset $\mathcal{S}$ of $\mathbb{B}(H)$ we call $\mathcal{S}''$ the von Neumann algebra generated by $\mathcal{S}$.

We now introduce several operator topologies on $\mathbb{B}(H)$. Denote by $\ell^2(H)$ the set of all sequences $\{\xi_n\}_{n=1}^{\infty} \subset H$ such that $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$ holds.

Definition 1.7. On $\mathbb{B}(H)$, we define the following topologies:

(1) The *strong operator topology* (SOT for short) is the locally convex topology determined by the seminorm family $\{p_\xi\}_{\xi \in H \setminus \{0\}}$, where

$$p_\xi(x) := \|x\xi\|, \quad x \in \mathbb{B}(H).$$

(2) The *weak operator topology* (WOT for short) is the locally convex topology determined by the seminorm family $\{p_{\xi,\eta}\}_{\xi,\eta \in H \setminus \{0\}}$, where

$$p_{\xi,\eta}(x) := |\langle x\xi, \eta\rangle|, \quad x \in \mathbb{B}(H).$$

(3) The *strong* operator topology* (S*OT for short) is the locally convex topology determined by the seminorm family $\{p_\xi\}_{\xi \in H \setminus \{0\}}$, where

$$p_\xi(x) := \|x\xi\| + \|x^*\xi\|, \quad x \in \mathbb{B}(H).$$

(4) The *σ-strong topology* (σSOT for short) is the locally convex topology determined by the seminorm family $\{p_{\{\xi_n\}}\}_{\{\xi_n\} \in \ell^2(H) \setminus \{0\}}$, where

$$p_{\{\xi_n\}}(x) := \left\{ \sum_{n=1}^{\infty} \|x\xi_n\|^2 \right\}^{\frac{1}{2}}, \quad x \in \mathbb{B}(H).$$

(5) The *σ-weak topolgy* (σWOT for short) is the locally convex topology determined by the seminorm family $\{p_{\{\xi_n\},\{\eta_n\}}\}_{\{\xi_n\},\{\eta_n\} \in \ell^2(H) \setminus \{0\}}$, where

$$p_{\{\xi_n\},\{\eta_n\}}(x) := \left| \sum_{n=1}^{\infty} \langle x\xi_n, \eta_n\rangle \right|, \quad x \in \mathbb{B}(H).$$

(6) The *σ-strong* topology* (σS*OT for short) is the locally convex topology determined by the seminorm family $\{p_{\{\xi_n\}}\}_{\{\xi_n\} \in \ell^2(H) \setminus \{0\}}$, where

$$p_{\{\xi_n\}}(x) := \left\{ \sum_{n=1}^{\infty} \left(\|x\xi_n\|^2 + \|x^*\xi_n\|^2 \right) \right\}^{\frac{1}{2}}, \quad x \in \mathbb{B}(H).$$

The product $(x, y) \mapsto xy$ is strongly continuous on the unit ball $\mathbb{B}(H)_1 := \{a \in \mathbb{B}(H); \|a\| \leq 1\}$, and on the unit ball, σ-strong topology coincides with strong topology.

Definition 1.8. Let M, N be von Neumann algebras. A linear map $\varphi : M \to N$ is called *normal* if it is σ-weakly continuous.

The notion of positivity is important for operator algebras. We recall the following fact from spectral theory.

Definition 1.9. For $x \in \mathbb{B}(H)$, we define the *spectrum* of x, denoted $\mathrm{sp}(x)$ by

$$\mathrm{sp}(x) := \{\lambda \in \mathbb{C};\ (x - \lambda 1) \text{ is not invertible in } \mathbb{B}(H)\}.$$

The spectrum $\mathrm{sp}(x)$ is always a nonempty compact subset of $\mathbb{C}$. And if x is normal (i.e., $xx^* = x^*x$), then x is self-adjoint (i.e., $x = x^*$) if and only if $\mathrm{sp}(x) \subset \mathbb{R}$ (this equivalence fails for non-normal x: consider nilpotent matrices!).

Proposition 1.10. *For $a = a^* \in \mathbb{B}(H)$, the following conditions are equivalent:*

(1) $\mathrm{sp}(a) \subset [0, \infty)$.

(2) $a = b^*b$ *for some* $b \in \mathbb{B}(H)$.

(3) $a = h^2$ *for a unique self-adjoint* $h \in \mathbb{B}(H)$ *with* $\mathrm{sp}(h) \subset [0, \infty)$.

(4) $\langle a\xi, \xi\rangle \geq 0$ *for every* $\xi \in H$.

If $A \subset \mathbb{B}(H)$ is a C*-*algebra and $a \in A$ then b (resp. h) in (2) (resp. in (3)) also belongs to A.*

Definition 1.11. If one (hence all) of the conditions (1)-(4) in Proposition 1.10 are satisfied for $a = a^* \in \mathbb{B}(H)$, we say that a is positive, and denote $a \geq 0$. The h in (3) is written as $a^{\frac{1}{2}}$. Also, for self-adjoint operators $a, b \in \mathbb{B}(H)$ we write $a \leq b$ if $b - a \geq 0$.

Lemma 1.12. *Let $A \subset \mathbb{B}(H)$ be a* C*-*algebra with unit 1_H. Then A is spanned by the unitary elements $\mathcal{U}(A) = \{u \in A; u^*u = 1 = uu^*\}$.*

Sketch of Proof. It suffices to show that every element $a \in A$ with norm $\|a\| \leq 1$ is spanned by unitaries in A. Since $a = a_1 + ia_2$, where $a_1 = \frac{a+a^*}{2}, a_2 = \frac{a-a^*}{2i}$ satisfy $a_i = a_i^*, \|a_i\| \leq 1 (i = 1, 2)$, we may from the beginning assume that $a = a^*$. In this case, $-1 \leq a \leq 1$ (in operator order) and therefore $1 - a^2 \geq 0$ (see Definition 1.11). In this case one can define the square root inside A: $\sqrt{1 - a^2} \in A$. Then

$$a = \frac{a + i\sqrt{1 - a^2}}{2} + \frac{a - i\sqrt{1 - a^2}}{2},$$

and it is not difficult to see that $a \pm i\sqrt{1 - a^2}$ are unitaries in A. □

Proposition 1.13. *Let $a, b \in \mathbb{B}(H)$. The following operator inequalities hold.*

(1) $b^*a^*ab \leq \|a\|^2 b^*b$.

(2) $(a + b)^*(a + b) \leq 2(a^*a + b^*b)$.

Proof. (1) Let $\xi \in H$. Then

$$\langle b^*a^*ab\xi, \xi\rangle = \langle ab\xi, ab\xi\rangle = \|ab\xi\|^2 \leq \|a\|^2\|b\xi\|^2 = \langle \|a\|^2 b^*b\xi, \xi\rangle.$$

Since $\xi \in H$ is arbitrary, we get (1).
(2) $2(a^*a + b^*b) - (a + b)^*(a + b) = a^*a + b^*b - a^*b - b^*a = (a - b)^*(a - b) \geq 0$. □

Definition 1.14. Let $1 \in A \subset \mathbb{B}(H)$ be a C*-algebra. A linear functional $\varphi \colon A \to \mathbb{C}$ is called

(1) *positive* if it satisfies $\varphi(a^*a) \geq 0$ for every $a \in A$.

(2) if moreover φ satisfies $\varphi(1) = 1$, it is called a *state*.

(3) A positive linear functional φ is called *faithful* if $\varphi(a^*a) = 0$ implies $a = 0$.

(4) A positive linear functional φ is called a *trace* if $\varphi(ab) = \varphi(ba)$ for $a, b \in A$.

Remark 1.15. If φ is a state on A, then its norm is attained at 1:

$$\|\varphi\| \stackrel{\mathrm{def}}{=} \sup\{|\varphi(a)|; a \in A, \|a\| \leq 1\} = \varphi(1) = 1.$$

We use next useful lemma later.

Lemma 1.16 (Cauchy-Schwartz inequality)**.** *Let A be a* C*-*algebra with unit on a Hilbert space H. Let φ be a state on A. Then for $a, b \in A$, the following inequality hold:*

$$|\varphi(b^*a)| \leq \varphi(b^*b)^{\frac{1}{2}} \varphi(a^*a)^{\frac{1}{2}}.$$

Moreover, we have $\|a + b\|_\varphi \leq \|a\|_\varphi + \|b\|_\varphi$ for every $a, b \in A$. Here, $\|a\|_\varphi := \varphi(a^*a)^{\frac{1}{2}}$.

Proof. **Step 1** $\varphi(a^*) = \overline{\varphi(a)}$ (complex conjugate) for every $a \in A$.
First we see that if $a = a^* \in A$, then $\varphi(a) \in \mathbb{R}$. This is because of the ineuqlity $-\|a\|1 \le a \le \|a\|1$, we have $\varphi(\|a\|1 - a) \ge 0$ and $\varphi(a + \|a\|1) \ge 0$. In particular, $\|a\| \ge \varphi(a)$ and $\varphi(a) \in \mathbb{R}$. Then for general $a \in A$, we have the decompositon $a = x + iy$ where $x = \frac{a+a^*}{2}, y = \frac{a-a^*}{2i} \in A$ are self-adjoint. Since $\varphi(x), \varphi(y) \in \mathbb{R}$, we know that

$$\varphi(a^*) = \varphi(x - iy) = \varphi(x) - i\varphi(y) = \overline{\varphi(x) + i\varphi(y)} = \overline{\varphi(a)}.$$

Step 2 We show the claim. It suffices to consider the case $\varphi(b^*b) > 0$. Since the transformatio $b \mapsto e^{i\theta}b$ ($\theta \in \mathbb{R}$) does not change $\varphi(b^*b)$, we may assume that $\varphi(b^*a) \in \mathbb{R}$ after this transformation. Now consider

$$(a + tb)^*(a + tb) = a^*a + t(a^*b + b^*a) + t^2b^*b \ge 0$$

for $t \in \mathbb{R}$. Applying φ to the above expansion we get that

$$\varphi(a^*a) + 2t\varphi(b^*a) + t^2\varphi(b^*b) \ge 0, \quad t \in \mathbb{R} \tag{2}$$

Here we used the fact (Step 1) that $\varphi(a^*b) = \varphi((b^*a)^*) = \overline{\varphi(b^*a)} = \varphi(b^*a) \in \mathbb{R}$. Since (2) is quadratic in t, (2) holds for every $t \in \mathbb{R}$ if and only if its discriminant is ≤ 0, whence

$$\varphi(b^*a)^2 - \varphi(a^*a)\varphi(b^*b) \le 0,$$

which shows (2).

We show the triangle inequality $\|a + b\|_\varphi \le \|a\|_\varphi + \|b\|_\varphi$.
Let $a, b \in A$. Then

$$\begin{aligned}(\|a\|_\varphi + \|b\|_\varphi)^2 - \|a + b\|_\varphi^2 &= (\|a\|_\varphi^2 + 2\|a\|_\varphi\|b\|_\varphi + \|b\|_\varphi^2) - (\|a\|_\varphi^2 + \varphi(a^*b + b^*a) + \|b\|_\varphi + \|b\|_\varphi^2)\\ &= 2(\varphi(a^*a)^{\frac{1}{2}}\varphi(b^*b)^{\frac{1}{2}} - \mathrm{Re}\varphi(a^*b))\\ &\ge 0,\end{aligned}$$

by the Cauchy-Schwartz inequality (and $\varphi(x^*) = \overline{\varphi(x)}$) that we have just proved. □

Now we give an (unofficial, but equivalent to the official) definition of II_1 factors. The official definition requires the analysis of projection lattice in von Neumann algebras (see e.g., [Su86]).

Definition 1.17. A von Neumann algebra $M \subset \mathbb{B}(H)$ is called a II_1 *factor*, if

(factor) The *center* $\mathcal{Z}(M)$ of M, $\mathcal{Z}(M) := M' \cap M$ is $\mathbb{C}1$.

(∞-dim) M is infinite-dimensional (as a vector space over $\mathbb{C}$).

(tracial) M admits a faithful normal tracial state τ.

We will not prove, but the following holds.

Proposition 1.18. *In a* II_1 *factor, there is only one normal faithful tracial state.*

Next theorem satets that we can construct a *-homomorphism of a C*-algbera out of states such that the resulting *-homomorphism is unique in a strong sense.

Theorem 1.19 (Gelfand-Naimark-Segal (GNS) construction)**.** *Let A be a* C*-*algebra with unit* 1, *and let φ be a state on A. Then there exists a triple $(H_\varphi, \pi_\varphi, \xi_\varphi)$ where H_φ is a Hilbert space, $\xi_\varphi \in H_\varphi$ is a unit vector and $\pi_\varphi \colon A \to \mathbb{B}(H_\varphi)$ is a *-homomorphism such that*

(1) $\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle$ ($a \in A$).

(2) $\pi_\varphi(A)\xi_\varphi = \{\pi_\varphi(a)\xi_\varphi; a \in A\}$ *is a dense subspace of* H_φ.

The triple is unique in the sense that if (H, ξ, π) is another triple satisfying (1) and (2), then there exists a unitary $V\colon H_\varphi \to H$ such that

$$V\pi_\varphi(a)V^* = \pi(a), \qquad a \in A,$$

and $V\xi_\varphi = \xi$.

Proof. Regard A as a pre-inner product space equipped with the following (possibly degenerate) inner-produt

$$\langle a, b\rangle_\varphi := \varphi(b^*a), \qquad a, b \in A.$$

It is clear that $\langle\,\cdot\,,\,\cdot\,\rangle_\varphi$ is sesqui-linear and positive (by the positivity of φ). Let $N_\varphi = \{a \in A; \|a\|_\varphi = \langle a, a\rangle_\varphi^{\frac{1}{2}} = 0\}$. Then N_φ is a vector subspace of A. It is clear that $x \in N_\varphi, \lambda \in \mathbb{C}$ implies $\lambda x \in N_\varphi$. If $x, y \in N_\varphi$, then by Lemma 1.16, $\|x + y\|_\varphi \le \|x\|_\varphi + \|y\|_\varphi = 0$, whence $x + y \in N_\varphi$. Therefore $\langle\,\cdot\,,\,\cdot\,\rangle_\varphi$ induces a positive definite inner-product on the quotient vector space A/N_φ. Let H_φ be the completion of A/N_φ and denote by $\hat{a}$ the canonical image of $a \in A$ in H_φ. By construction, $\widehat{A} = \{\hat{a}; a \in A\}$ is a dense subspace of H_φ. Define $\xi_\varphi = \hat{1} \in H_\varphi$. Then $\|\xi_\varphi\|^2 = \langle 1, 1\rangle_\varphi = \varphi(1) = 1$. So ξ_φ is a unit vector. For each $a \in A$, define $\pi_\varphi(a)$ as a linear map $\widehat{A} \to \widehat{A}$ by

$$\pi_\varphi(a)\hat{b} = \widehat{ab}, \quad b \in A.$$

Since $b^*a^*ab \le \|a\|^2 b^*b$ by Lemma 1, we see that $\pi_\varphi(a)$ is bounded on the dense subspace $\widehat{A}$ of H_φ:

$$\|\widehat{ab}\|^2 = \varphi(b^*a^*ab) \le \|a\|^2\varphi(b^*b) = \|a\|^2\|\hat{b}\|^2.$$

Therefore $\pi_\varphi(a)$ is uniquely extended to a bounded linear map $\pi_\varphi(a) \in \mathbb{B}(H_\varphi)$. It is straightforward to see that $A \ni a \mapsto \pi_\varphi(a) \in \mathbb{B}(H_\varphi)$ is a *-homomorphism. For example, if $a, b, c \in A$. Then

$$\pi_\varphi(a)\pi_\varphi(b)\hat{c} = \widehat{abc} = \pi_\varphi(ab)\hat{c}.$$

Therefore $\pi_\varphi(a)\pi_\varphi(b)$ and $\pi_\varphi(ab)$ agree on the dense subspace $\widehat{A}$ of H_φ, whence by boundedness, $\pi_\varphi(a)\pi_\varphi(b) = \pi_\varphi(ab)$ as operator in $\mathbb{B}(H_\varphi)$. Next, if $a \in A$, then

$$\langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi\rangle = \langle \hat{a}, \hat{1}\rangle_\varphi = \varphi(a).$$

It is clear that $\pi_\varphi(A)\xi_\varphi = \widehat{A}$, which is dense in H_φ.

Finally, suppose (H, π, ξ) is another triple satisfying (1) (2). Then let $V\colon \pi_\varphi(A)\xi_\varphi = \widehat{A} \to \pi(A)\xi$ by

$$V\pi_\varphi(a)\xi_\varphi := \pi(a)\xi.$$

This is well-defined, because $\|\pi_\varphi(a)\xi_\varphi\|^2 = \varphi(a^*a) = \langle \pi(a^*a)\xi, \xi\rangle = \|\pi(a)\xi\|^2$, and V is in fact isometric on $\widehat{A}$. Therefore V is extended to an isometry from H_φ into H. If we also define $W\colon H \to H_\varphi$ as the unique isometric extension of the map $\pi(A)\xi \to \pi_\varphi(A)\xi_\varphi$ given by $W\pi(a)\xi = \pi_\varphi(a)\xi_\varphi$, then it is straightforward to see that $VW = 1_H, WV = 1_{H_\varphi}$ (again use densities of $\widehat{A}, \pi(A)\xi$ in H_φ, H, respectively). Therefore V is onto, and it is clear that $V\xi_\varphi = \xi$ by construction. Now let $a, b, c \in A$. Then

$$\langle V\pi_\varphi(a)\pi_\varphi(b)\xi_\varphi, \pi(c)\xi\rangle = \langle V\pi_\varphi(ab)\xi_\varphi, \pi(c)\xi\rangle = \langle \pi(ab)\xi, \pi(c)\xi\rangle,$$
$$\langle \pi(a)V\pi_\varphi(b)\xi_\varphi, \pi(c)\xi\rangle = \langle \pi(a)\pi(b)\xi, \pi(c)\xi\rangle = \langle \pi(ab)\xi, \pi(c)\xi\rangle.$$

Therefore since both $V\pi_\varphi(a), \pi(a)V$ are bounded operators, the density of $\pi_\varphi(A)\xi_\varphi$ (resp. $\pi(A)\xi$) in H_φ (resp. in H) shows that $V\pi_\varphi(a) = \pi(a)V$. This finishes the proof. ☐

Now we give an example of a hyperfinite II_1 factor.

Definition 1.20. A von Neumann algebra M is called *hyperfinite* if there exists an increasing sequence $M_1 \subset M_2 \subset \cdots$ of finite-dimensional *-subalgebra of M with unit 1, such that its union is SOT-dense in M: $M = \overline{\bigcup_{n\in\mathbb{N}} M_n}^{\mathrm{SOT}}$.

Example 1.21 (Hyperfinite II_1 factor)**.** Consider an increasing sequence of matrix algebras

$$M_2(\mathbb{C}) \hookrightarrow M_2(\mathbb{C})^{\otimes 2} \hookrightarrow M_2(\mathbb{C})^{\otimes 3} \hookrightarrow \cdots,$$

where the inclusion $M_2(\mathbb{C})^{\otimes n} \hookrightarrow M_2(\mathbb{C})^{\otimes(n+1)}$ is given by $x \mapsto x \otimes 1$ for each $n \in \mathbb{N}$. Let M_{2^∞} be the norm-completion of the above inductive limit $\bigcup_{n\in\mathbb{N}} M_2(\mathbb{C})^{\otimes n}$. Consider for each $n \in \mathbb{N}$ a tracial state $\tau_n = (\frac{1}{2}\mathrm{Tr})^{\otimes n}$ on $M_2(\mathbb{C})^{\otimes n} \cong M_{2^n}(\mathbb{C})$. Namely τ_n is uniquely determined (by the universality of tensor product)

$$\tau_n(x_1 \otimes \cdots \otimes x_n) = \frac{1}{2}\mathrm{Tr}(x_1)\cdots\frac{1}{2}\mathrm{Tr}(x_n), \qquad x_1, x_2, \cdots, x_n \in M_2(\mathbb{C}).$$

Since $\tau_n(y) = \tau_{n+1}(y \otimes 1)$ ($y \in M_2(\mathbb{C})^{\otimes n}$), and since states are norm-continuous, $\{\tau_n\}_{n=1}^\infty$ defines a positive linear functional (of norm 1) on $\bigcup_{n\in\mathbb{N}} M_2(\mathbb{C})^{\otimes n}$ which is extended to a unique state τ on the (abstract) C*-algebra $M_{2^\infty} = \overline{\bigcup_{\mathbb{N}} M_2(\mathbb{C})^{\otimes n}}^{\|\cdot\|}$. Let $(H_\tau, \xi_\tau, \pi_\tau)$ be the GNS representation associated with τ. We then define

$$R := \pi_\tau(M_{2^\infty})''$$

τ is extended to a normal tracial state by

$$\tau(x) = \langle x\xi_\tau, \xi_\tau\rangle, \qquad x \in \pi_\tau(M_{2^\infty})'' = R.$$

It is clear from the construction that R is hyperfinite. Below we give a proof that R is indeed a II_1 factor.

We show that R is a hyperfinite II_1 factor. It is clear that R is hyperfinite, infinite-dimensional and has a normal tracial state τ. We show that

(a) τ is faithful on R.

(b) R is a factor.

For (a), note that we do know that τ is faithful on a strongly dense *-subalgebra $\bigcup_{\mathbb{N}} M_2(\mathbb{C})^{\otimes n}$. However, this does not automatically mean that τ is faithful on R (and in fact could fail if we consider non-tracial states). We deduce (b) from

(b') R has unique normal tracial state.

Lemma 1.22. *Let M be a von Neumann algebra on a Hilbert space H. Assume that there exists a cyclic vector $\xi \in H$ for M such that $\tau(x) = \langle x\xi, \xi\rangle$ defines a tracial state on M. Then τ is faithful.*

Proof. Let $x \in M$ be such that $\tau(x^*x) = 0$, and let $a \in M$. Then since τ is tracial, we have (use Propositin 1.13)

$$\begin{aligned}\|xa\xi\|^2 &= \tau(a^*x^*xa) = \tau(xaa^*x^*)\\ &\le \|a^*\|^2\tau(xx^*)^2 = \|a\|^2\tau(x^*x)\\ &= 0.\end{aligned}$$

Therefore $xa\xi = 0$. Since ξ is cyclic for M, $\{a\xi; a \in M\}$ is dense in H, whence $x = 0$ holds. □

Proof of (a). Applying Lemma 1.22 to $M = R$, we get that $\tau = \langle\,\cdot\,\xi_\tau, \xi_\tau\rangle$ is a faithful normal tracial state on R. □

We go to the proof of (b'). This follows from the next easy lemma:

Lemma 1.23. *Let $n \in \mathbb{N}$. Then $\frac{1}{n}\mathrm{Tr}(\cdot)$ is the only tracial state on $M_n(\mathbb{C})$.*

Proof. Let τ be a tracial state on $M_n(\mathbb{C})$. Let $\{e_{ij}; 1 \le i,j \le n\}$ be the standard system of matrix units in $M_n(\mathbb{C})$. That is, the (k,l)-component of e_{ij} is 1 precisely when $(k,l) = (i,j)$ and 0 otherwise $(1 \le k,l \le n)$. It is easy to see that they satisfy

$$e_{ij}e_{kl} = \delta_{jk}e_{il}, \quad \sum_{i=1}^{n} e_{ii} = 1, \qquad 1 \le i,j,k,l \le n.$$

Then we show that for each $1 \le i,j \le n$, we have

$$\tau(e_{ij}) = \frac{1}{n}\delta_{ij}.$$

Indeed, for each $1 \le i,j \le n$, it holds that (use the trace property of τ)

$$\tau(e_{ii}) = \tau(e_{ij}e_{ji}) = \tau(e_{ji}e_{ij}) = \tau(e_{jj}).$$

Therefore since $e_{11} + \cdots + e_{nn} = 1$,

$$1 = \sum_{k=1}^{n} \tau(e_{kk}) = n\tau(e_{ii}), \quad 1 \le i \le n.$$

On the other hand, if $i \ne j$, then

$$\tau(e_{ij}) = \tau(e_{ij}e_{jj}) = \tau(e_{jj}e_{ij}) = \tau(\delta_{ji}e_{jj}) = 0.$$

This shows that since each $x \in M_n(\mathbb{C})$ is uniquely written as $x = \sum_{i,j=1}^{n} x_{ij}e_{ij}$ $(x_{ij} \in \mathbb{C})$,

$$\tau(x) = \sum_{i,j=1}^{n} x_{ij}\tau(e_{ij}) = \frac{1}{n}\sum_{i=1}^{n} x_{ii} = \frac{1}{n}\mathrm{Tr}(x).$$

This shows that $\tau = \frac{1}{n}\mathrm{Tr}$. □

Proof of (b'): R has unique normal faithful tracial state. Let τ' be a normal tracial state on R. We show that $\tau = \tau'$. Since τ, τ' are normal, it suffices to show that τ and τ' agree on the strongly dense *-subalgebra $R_0 = \bigcup_{n\in\mathbb{N}} M_2(\mathbb{C})^{\otimes n}$. But by Lemma 1.23, there is only one tracial state on $M_2(\mathbb{C})^{\otimes n} \cong M_{2^n}(\mathbb{C})$. Therefore $\tau|_{M_2(\mathbb{C})^{\otimes n}} = \tau'|_{M_2(\mathbb{C})^{\otimes n}}$ for every $n \in \mathbb{N}$, which finishes the proof. □

Proof of (b): R is a factor. By spectral theory (cf. Corollary 1.5)[1], it can be shown that the center $\mathcal{Z}(R)$ is generated by positive elements. So we have only to show that $\mathcal{Z}(R)_+ = \mathbb{R}_+1$. Let $h \in \mathcal{Z}(R)_+$ be a nonzero positive element. By rescaling we may assume that $\tau(h) = 1$, since τ is faithful by (a). Then since h is in the center, the new normal state $\tau' = \tau(h\cdot) = \langle \cdot\, \xi_\tau, h\xi_\tau\rangle$ is tracial (and normal):

$$\tau'(xy) = \tau(hxy) = \tau(xhy) \overset{\text{trace}}{=} \tau(hyx) = \tau'(yx) \ (x,y \in R).$$

Therefore $\tau' = \tau$ by (b'). This means that for every $x \in R$, one has

$$\tau(hx) = \tau(x) \Leftrightarrow \langle \hat{x}, \hat{h}\rangle = \langle \hat{x}, \hat{1}\rangle.$$

Since $\{\hat{x}; x \in R\}$ is dense in the GNS Hilbert space $L^2(R,\tau)$, $\hat{h} = \hat{1}$, which shows that $h = 1$. Therefore $\mathcal{Z}(R)_+ = \mathbb{R}_+1$, and we are done. □

Thus we finally see that R is a hyperfinite II_1 factor.

Theorem 1.24 (Murray-von Neumann)**.** *Every hyperfinite* II_1 *factor on a separable and infinite-dimensional Hilbert space is *-isomorphic to R.*

[1] $\mathcal{Z}(R)$ is spanned by its self-adjoint elements. Decompose a self-adjoint element $x = x^* \in \mathcal{Z}(R)$ as a difference of two positive elements $x = x_+ - x_-$, $x_+x_- = 0$ using spectral decomposition. Then with a more argument using Corollary 1.5 shows that $x_\pm \in \mathcal{Z}(R)_+$.

Hence R is *the* hyperfinite II_1 factor.

Remark 1.25. (1) It is a nontrivial fact, that on a II_1 *factor* M, every *tracial* state is automatically normal (and faithful). This does not hold if we drop either a factoriality of M or tracial condition.

(2) It is a much more important and nontrivial fact, that a von Neumann subalgebra of the hyperfinite von Neumann algebra R is hyperfinite (see [Co76]). Therefore II_1 subfactors of R are all isomorphic to R itself. This does *not* follow from Murray-von Neumann's uniqueness theorem.

2 Talk 2 (Oct. 8 2014, 10:30-11:15)

We cover the following two topics.

(A) Standard representation

(B) Conditional expectation

(A) Standard representation

Let M be a von Neumann algebra with a faithful normal tracial state τ. Let $(L^2(M,\tau) = L^2(M), \xi_\tau, \pi_\tau)$ be the GNS representation of τ. Namely, (since τ is faithful, $N_\tau = \{x \in M; \tau(x^*x) = 0\} = \{0\}$), $L^2(M)$ is the completion of M with respect to

$$\langle a, b\rangle_\tau = \tau(b^*a), \qquad a, b \in M.$$

The canonical image of $a \in M$ in $L^2(M)$ is denoted as $\hat{a} \in L^2(M)$, and by definition $\widehat{M} = \{\hat{a}; a \in M\}$ is a dense subspace of $L^2(M)$. For each $a \in M$, $\pi(a)$ is the uniquely bounded operator in $L^2(M)$ determined by the relation

$$\pi_\tau(a)\hat{b} = \widehat{ab}, \qquad b \in M.$$

$M \ni a \mapsto \pi_\tau(a) \in \mathbb{B}(L^2(M))$ is an injective *-homomorphism. To see the injectivity, let $\pi_\tau(a) = 0$. Then $0 = \|\pi_\tau(a)\xi_\tau\|^2 = \tau(a^*a)$, whence $a = 0$ by the faithfulness of τ.

It takes little more work to show that (proof omitted)

Proposition 2.1. *π_τ is an injective normal *-homomorphism, and $\pi_\tau(M)$ is a von Neumann algebra on $L^2(M)$ (i.e., $\pi_\tau(M)$ is* SOT*-closed).*

Therefore we identify M with $\pi_\tau(M)$ and regard $M \subset \mathbb{B}(L^2(M))$. Because of this identification, it is very important to understand whether we regard $a \in M$ as an operator $a \in \mathbb{B}(L^2(M)$ or as a vector $\hat{a} \in L^2(M)$. From now on, in order to avoid confusion, we use $\|\cdot\|_\infty$ to denote the operator norm (as a bounded operator on $L^2(M)$) and $\|\cdot\|_2$ to denote the Hilbert space norm on $L^2(M)$. Therefore for $a \in M$:

$$\|a\|_\infty = \sup_{\|\xi\|_2 \le 1} \|a\xi\|_2, \quad \|\hat{a}\|_2 = \tau(a^*a)^{\frac{1}{2}}.$$

In general, it is a nontrivial task to compute the commutant M' of a given von Neumann algebra. However, when M is represented in the special Hilbert space $L^2(M)$, the commutant is explicitly described. To describe the commutant, let J be an operator $\widehat{M} \to \widehat{M}$ defined as

$$J\hat{a} := \widehat{a^*}, \qquad a \in M.$$

Note that $\|\widehat{a^*}\|_2^2 = \tau(aa^*) = \tau(a^*a) = \|\hat{a}\|_2^2$, whence J is isometric on the dense subspace $\widehat{M}$ of $L^2(M)$. Therefore J is extended to a conjugate-linear isometry $J\colon L^2(M) \to L^2(M)$ by continuity satisfying $J \circ J = \mathrm{id}_{L^2(M)}$ (in particular, J is invertible). First we see that

Lemma 2.2. *$JMJ \subset M'$.*

Proof. Let $a, b, c \in M$. Then

$$\begin{aligned} JaJb\hat{c} &= Ja\widehat{c^*b^*} = J\widehat{ac^*b^*} = \widehat{bca^*}, \\ bJaJ\hat{c} &= bJa\widehat{c^*} = b\widehat{ca^*} = \widehat{bca^*}. \end{aligned}$$

This shows that $JaJb$ and $bJaJ$ agree on the dense subspace $\widehat{M}$ of $L^2(M)$. Since they are both bounded, this shows that $JaJb = bJaJ$. Since $b \in M$ is arbitrary, we have $JaJ \in M'$. Therefore $JMJ \subset M'$. □

Actually, the opposite inclusion holds:

Theorem 2.3. $JMJ = M'$.

Proof. The proof goes in 3 steps.
Step 1. Let $x' \in M'$. Then $Jx'\xi_\tau = (x')^*\xi_\tau$.
Given $a \in M$, we have (use $x'a = ax'$)

$$\begin{aligned} \langle Jx'\xi_\tau, \hat{a}\rangle &= \langle J\hat{a}, x'\xi_\tau\rangle = \langle \widehat{a^*}, x'\xi_\tau\rangle \\ &= \langle (x')^*a^*\xi_\tau, \xi_\tau\rangle = \langle a^*(x')^*\xi_\tau, \xi_\tau\rangle \\ &= \langle (x')^*\xi_\tau, a\xi_\tau\rangle = \langle (x')^*\xi_\tau, \hat{a}\rangle. \end{aligned}$$

Since $\widehat{M}$ is dense in $L^2(M)$, we have $Jx'\xi_\tau = (x')^*\xi_\tau$.

Step 2. $M'\xi_\tau = \{x'\xi_\tau ; x' \in M'\}$ is dense in $L^2(M)$.
Let $E = \overline{M'\xi_\tau}$. Then E is a closed subspace of $L^2(M)$. Let $p = P_E$ be the orthogonal projection of $L^2(M)$ onto E. It suffices to show that $p = 1$ in order to prove the density. We first see that E is invariant under M', whence $E^\perp$ is also M'-invariant. This shows that p commutes with every $a' \in M'$. Therefore by double commutant theorem, we have $p \in (M')' = M$. Since $1 \in M'$, we know that $\xi_\tau \in E$, and $\xi_\tau = p\xi_\tau$. Therefore by $p \in M$, it holds that

$$0 = \|(1-p)\xi_\tau\|_2^2 = \tau((1-p)^*(1-p)).$$

This shows that $1 - p = 0$ by the faithfulness of τ, as desired.

Step 3. $M' \subset JMJ$.
Let $x', y', z' \in M'$. Then by Step 1, we have

$$\begin{aligned} Jx'Jy'(z'\xi_\tau) &= Jx'(z')^*(y')^*\xi_\tau = y'z'(x')^*\xi_\tau \\ &= y'Jx'(z')^*\xi_\tau = y'Jx'J(z'\xi_\tau). \end{aligned}$$

By Step 2, $M'\xi_\tau$ is dense. Therefore by boundedness of involved operators, $Jx'Jy' = y'Jx'J$ holds. Since $y' \in M'$ is arbitrary, we see (again by double commutant theorem) that

$$Jx'J \in (M')' = M \Leftrightarrow x' \in JMJ.$$

Therefore $M' \subset JMJ$. Together with Lemma 2.2, we have shown that $JMJ = M'$. □

As a consequence: we know by construction that M acts on $L^2(M)$ from the left by operator multiplication. On the other hand, we can also define the right M-action on $L^2(M)$ by

$$\xi \cdot x := Jx^*J\xi, \qquad x \in M, \ \xi \in L^2(M)$$

By Theorem 2.3, these actions commute:

$$a \cdot (\xi \cdot b) = (a \cdot \xi) \cdot b, \qquad a, b \in M, \ \xi \in L^2(M),$$

and they are the commutant of each other. It is (probably) important for Prof. Nest's talks tomorrow to regard $L^2(M)$ as an M-M bimodule. Finally, let us see that any normal state on $M \subset \mathbb{B}(L^2(M))$ is a vector state:

Theorem 2.4. *Let φ be a normal state on $M \subset \mathbb{B}(L^2(M))$. Then there is a vector $\xi_\varphi \in L^2(M)_+ = \overline{M_+}^{\|\cdot\|_2}$ such that $\varphi = \langle \cdot \xi_\varphi, \xi_\varphi\rangle$.*

Unfortunately we need the following difficult result:

Theorem 2.5 (Powers-Størmer inequality)**.** *Let $\xi, \eta \in L^2(M)_+$. Then if we let $\omega_\zeta = \langle \cdot \zeta, \zeta\rangle$ ($\zeta \in L^2(M)_+$), one has*

$$\|\xi - \eta\|_2^2 \leq \|\omega_\xi - \omega_\eta\| \leq \|\xi - \eta\|_2 \|\xi + \eta\|_2.$$

Proof of Theorem 2.4 assuming Theorem 2.5. Since φ is σ-weakly continuous, there exists a sequence $(\xi_n)_{n=1}^\infty \in \ell^2(L^2(M))$ with $\sum_{n=1}^\infty \|\xi_n\|^2 = \varphi(1) = 1$ such that

$$\varphi(x) = \sum_{n=1}^\infty \langle x\xi_n, \xi_n\rangle, \qquad x \in M.$$

Step 1. For every $\varepsilon > 0$, there exists $b \in M_+$ such that $\|\varphi - \omega_{\hat{b}}\| < \varepsilon$.
Fix $\varepsilon > 0$. Let $\varphi_k(x) = \sum_{n=1}^k \langle x\xi_n, \xi_n\rangle$ ($x \in M, k \in \mathbb{N}$). Then since positive linear functional attains its norm at 1,

$$\|\varphi - \varphi_k\| = \sum_{n=k+1}^\infty \|\xi_n\|^2 \overset{k\to\infty}{\to} 0.$$

Then fix $k_0 \in \mathbb{N}$ such that $\|\varphi - \varphi_{k_0}\| < \varepsilon/2$. Since $\widehat{M} \subset L^2(M)$, there exists $a_n \in M$ ($1 \leq n \leq k_0$) such that $\|\widehat{a_n} - \xi_n\|_2 \|\widehat{a_n} + \xi_n\|_2 < \varepsilon/(2k_0)$. Then by Theorem 2.5, we have

$$\begin{aligned}\left\|\varphi_{k_0} - \sum_{n=1}^{k_0} \omega_{\widehat{a_n}}\right\| &\leq \sum_{n=1}^{k_0} \|\omega_{\xi_n} - \omega_{\widehat{a_n}}\| \\ &\leq \sum_{n=1}^{k_0} \|\widehat{a_n} - \xi_n\|_2 \|\widehat{a_n} + \xi_n\|_2 \\ &< \varepsilon/2.\end{aligned}$$

Also, for $x \in M$,

$$\begin{aligned}\sum_{n=1}^{k_0} \omega_{\widehat{a_n}}(x) &= \sum_{n=1}^{k_0} \tau(a_n^* x a_n) = \tau\left(\sum_{n=1}^{k_0} a_n a_n^* x\right) \\ &= \tau(bxb) = \omega_{\hat{b}}(x),\end{aligned}$$

where $b := \left(\sum_{n=1}^{k_0} a_n a_n^*\right)^{\frac{1}{2}} \in M_+$. Therefore $\|\varphi - \omega_{\hat{b}}\| < \varepsilon$.
Step 2. By Step 1, there exists $(b_n)_{n=1}^\infty \subset M_+$ such that $\|\varphi - \omega_{\widehat{b_n}}\| \to 0$ ($n \to \infty$). By Theorem 2.5, we have

$$\|\widehat{b_n} - \widehat{b_m}\|_2^2 \leq \|\omega_{\widehat{b_n}} - \omega_{\widehat{b_m}}\| \overset{n,m\to\infty}{\to} 0.$$

Therefore $(\widehat{b_n})_{n=1}^\infty$ is a Cauchy sequence in $L^2(M)_+$, whence has a limit

$$\xi_\varphi = \lim_{n\to\infty} \widehat{b_n} \in L^2(M)_+.$$

Again by Theorem 2.5,

$$\begin{aligned}\|\omega_{\xi_\varphi} - \varphi\| &= \lim_{n\to\infty} \|\omega_{\xi_\varphi} - \omega_{\widehat{b_n}}\| \\ &\leq \lim_{n\to\infty} \|\xi_\varphi - \widehat{b_n}\|_2 \|\xi_\varphi + \widehat{b_n}\|_2 \\ &= 0,\end{aligned}$$

whence $\varphi = \omega_\varphi$. □

(B) Conditional Expectations

Let $N \subset M$ be an inclusion of von Neumann algberas, and let τ be a normal faithful tracial state on M. As in (A), we let M act on the standard Hilbert space $L^2(M)$ by left multiplication (hence N also acts on $L^2(M)$). Let $\mathcal{H} = \overline{N\xi_\tau}$. Then N acts on $\mathcal{H}$ and $\xi_\tau \in \mathcal{H}$ is cyclic for the N-action. Moreover, $\tau|_N(x) = \langle x\xi_\tau, \xi_\tau\rangle$ for $x \in N$. Therefore by the uniqueness of the GNS construction, Theorem 1.19, we may identify $\mathcal{H} = L^2(N, \tau|_N) = L^2(N)$ and therefore we have an inclusion $L^2(N) \subset L^2(M)$.

Definition 2.6. We define e_N to be the orthogonal projection of $L^2(M)$ onto $L^2(N)$.

Theorem 2.7 (Trace-preserving conditional expectation)**.** *There exists a positive linear map $E_N \colon M \to N$ satisfying*

(1) $e_N(\hat{x}) = \widehat{E_N(x)}, \quad x \in M.$

(2) $E_N(axb) = aE_N(x)b, \quad a, b \in N, x \in M.$

(3) $\tau(E_N(x)) = \tau(x), \quad x \in M.$

(4) $e_N x e_N = E_N(x)e_N, \quad x \in M.$

E_N is faithful and normal.

Remark 2.8. A positive linear map $E\colon M \to N$ satisfying (2)(3) is unique $= E_N$.

Definition 2.9. E_N in Theorem 2.7 is called the *trace-preserving conditional expectation* of M onto N with respect to τ.

E_N will (probably) play an important role in Prof. Nest's talks. In order to construct E_N, we use the following useful lemma. Recall that M acts on $L^2(M)$ from the right by $\xi \cdot x = Jx^*J\xi$. We can also think of this as an "left action of ξ on $\hat{x}$" (though ξ is a vector, not an operator).

Lemma 2.10. *Let M be a von Neumann algebra with a faithful normal tracial state. Then for $\xi \in L^2(M)$, the following two conditions are equivalent.*

(1) *ξ is left-bounded. That is, there exists a constant $c > 0$ such that*

$$\|\xi \cdot x\|_2 \le c\|\hat{x}\|_2, \qquad x \in M.$$

(2) *$\xi \in \widehat{M}$. That is, there exists $a \in M$ such that $\xi = \hat{a}$.*

Proof. (2)⇒(1) Let $\xi = \hat{a} \in \widehat{M}$ ($a \in M$). Then for $x \in M$, we have

$$\begin{aligned}\|\xi \cdot x\|_2 &= \|Jx^*Ja\hat{\xi_\tau}\|_2 = \|aJx^*J\xi_\tau\|_2 = \|a\hat{x}\|_2 \\ &\le \|a\|_\infty \cdot \|\hat{x}\|_2,\end{aligned}$$

whence (1) holds with $c = \|a\|$ (operator norm of a).
(1)⇒(2) Assume that (1) holds. Then the linear map $L_\xi \colon \widehat{M} \to L^2(M)$ given by

$$L_\xi(\hat{x}) := \xi \cdot x, \qquad x \in M$$

extends to a bounded linear map $L_\xi \in \mathbb{B}(L^2(M))$ (with operator norm $\|L_\xi\|_\infty \le c$). We show that $a := L_\xi \in M$. To see this, let $x, y \in M$. Then

$$L_\xi JxJ\hat{y} = L_\xi\widehat{yx^*} = Jxy^*J\xi = JxJ(Jy^*J\xi) = JxJL_\xi\hat{y}.$$

Since L_ξ, JxJ are both bounded, density of $\widehat{M}$ in $L^2(M)$ implies that L_ξ and JxJ commute. By Theorem 2.3 and double commutant theorem, we see that

$$a = L_\xi \in (JMJ)' = (M')' = M.$$

Therefore

$$\xi = L_\xi(\hat{1}) = \hat{a} \in \widehat{M}.$$

This finishes the proof. □

Proof of Theorem 2.7. (1): In view of Lemma 2.10, we show that $e_N(\hat{x})$ is a left-bounded vector in $L^2(N)$. Let $y, z \in N$. Then

$$\begin{aligned}\langle Jy^*Je_N(\hat{x}), \hat{z}\rangle_{L^2(N)} &= \langle e_N(\hat{x}), JyJ\hat{z}\rangle_{L^2(N)} = \langle \hat{x}, e_N(\underbrace{\widehat{zy^*}}_{\in \widehat{N}\subset L^2(N)})\rangle \\ &= \langle \hat{x}, \widehat{zy^*}\rangle_{L^2(M)} = \langle \hat{x}, JyJ\hat{z}\rangle_{L^2(M)} \\ &= \langle Jy^*J\hat{x}, \hat{z}\rangle_{L^2(M)} = \langle \widehat{xy}, \hat{z}\rangle_{L^2(M)}\end{aligned}$$

Therefore

$$|\langle Jy^*Je_N(\hat{x}), \hat{z}\rangle_{L^2(N)}| \le \|x\hat{y}\|_2\|\hat{z}\|_2 \le \|x\|_\infty\|\hat{y}\|_2\|\hat{z}\|_2.$$

This shows that

$$\|Jy^*Je_N(\hat{x})\|_2 \le \|x\|_\infty\|\hat{y}\|_2.$$

Hence $e_N(\hat{x})$ is left-bounded. By Lemma 2.10, there exists $E_N(x) \in N$ such that $e_N(\hat{x}) = \widehat{E_N(x)}$. It is clear that $E_N\colon M \to N$ is linear. We prove the positivity of E_N later.
(2) We first note that J preserves $L^2(N)$, and $Je_N = e_NJ$. Therefore for each $x \in M$, we have

$$Je_N(\hat{x}) = e_NJ\hat{x} \Leftrightarrow \widehat{E_N(x)^*} = \widehat{E_N(x^*)}.$$

Therefore

$$E_N(x)^* = E_N(x^*). \tag{3}$$

We next show see that $E_N(ax) = aE_N(x)$ for $a \in N$ and $x \in M$. To see this, let $b \in N$. Then

$$\begin{aligned}\langle \widehat{E_N(ax)}, \hat{b}\rangle_{L^2(N)} &= \langle e_N(\widehat{ax}), \hat{b}\rangle_{L^2(N)} = \langle a\hat{x}, \underbrace{e_N(\hat{b})}_{=\hat{b}}\rangle_{L^2(M)} \\ &= \langle \hat{x}, \widehat{a^*b}\rangle_{L^2(M)} = \langle \hat{x}, e_N(\widehat{a^*b})\rangle_{L^2(M)} \\ &= \langle e_N(\hat{x}), \widehat{a^*b}\rangle_{L^2(N)} = \langle ae_N\hat{x}, \hat{b}\rangle_{L^2(N)} \\ &= \langle \widehat{aE_N(x)}, \hat{b}\rangle_{L^2(N)}.\end{aligned}$$

Therefore by the density of $\widehat{N}$ in $L^2(N)$, $E_N(ax) = aE_N(x)$ hold. Replacing a by b^* ($b \in N$) and x by x^* and use (3) to obtain $E_N(xb) = E_N(x)b$. This shows (2).
(3): This is straightforward, since for every $x \in M$,

$$\begin{aligned}\tau(E_N(x)) &= \langle \widehat{E_N(x)}, \hat{1}\rangle = \langle e_N(\hat{x}), \hat{1}\rangle \\ &= \langle \hat{x}, e_N(\hat{1})\rangle = \langle \hat{x}, \hat{1}\rangle \\ &= \tau(x).\end{aligned}$$

(4): Let $x, y \in M$. Then by (2), we have (use $b = E_N(y) \in N$)

$$\begin{aligned}e_Nxe_N\hat{y} &= e_Nx\widehat{E_N(y)} = e_N\widehat{xE_N(y)} = E_N\widehat{(xE_N(y))} \\ &= E_N\widehat{(x)E_N}(y) = E_N(x)e_N(\hat{y}).\end{aligned}$$

Therefore by the density of $\widehat{M}$ in $L^2(M)$, $e_Nxe_N = E_N(x)e_N$ holds.
Finally, we show that E_N preserves positivity. Let $x \in M_+$. Then for each $y \in N$,

$$\langle E_N(x)\hat{y}, \hat{y}\rangle = \langle y^*E_N(x)y\xi_\tau, \xi_\tau\rangle = \langle E_N(y^*xy)\xi_\tau, \xi_\tau\rangle = \tau(E_N(y^*xy)) \stackrel{(3)}{=} \tau(y^*xy) \ge 0.$$

Since $y \in N$ is arbitrary, the density of $\widehat{N}$ in $L^2(N)$ shows that $\langle E_N(x)\xi, \xi\rangle \ge 0$ holds for $\xi \in L^2(N)$, whence $E_N(x) \in N_+$. Let $x \in M$ be such that $E_N(x^*x) = 0$. Then by (3),

$$\tau(x^*x) = \tau(E_N(x^*x)) = 0.$$

This shows that $x = 0$, by the faithfulness of τ. Normality of E_N requires more effort, but it follows from the fact that E_N preserves the normal faithful trace $\tau \circ E_N = \tau$ (we do not discuss it here). $\square$

Definition 2.11. The von Neuman algebra on $L^2(M)$ generated by M and $\{e_N\}$ is denoted as $\langle M, e_N\rangle$, called the *Jones' basic construction*, and e_N is called *Jones projection*.

It can be checked directly, that $N = M \cap \{e_N\}'$ and $\langle M, e_N\rangle = JN'J$ (but we have no time to discuss it here!).

Appendix: $\dim_M H$ (Oct. 9 2014)

Let us add one more tool, the *von Neumann dimension* $\dim_M H$, that will be needed but could not be covered in the last two talks. Since we need the notion of semifinite trace (which we have not defined), this appendix is not entirely self-contained. Let M be von Neumann algebra with a normal faithful tracial state τ. We consider the representation theory of M. We denote the *opposite algebra* of M as M^{op}. That is, $M^{\mathrm{op}} = \{x^{\mathrm{op}}; x \in M\}$ is a copy of M as a set, equipped with the same linear space and *-operation, but the product is reversed:

$$x^{\mathrm{op}}y^{\mathrm{op}} = (yx)^{\mathrm{op}}, \quad x^{\mathrm{op}}, y^{\mathrm{op}} \in M^{\mathrm{op}}.$$

Definition 2.12. A *left M-module* is a Hilbert space H with the left M-action. That is, H is equipped with a normal *-homomorphism $\pi\colon M \to \mathbb{B}(H)$ and the left M-action is given by

$$x \cdot \xi := \pi(x)\xi, \quad x \in M, \xi \in H.$$

Similarly, a *right M-module* is a Hilbert space K with the right M-action, i.e, a normal *-homomorphism $\rho\colon M^{\mathrm{op}} \to \mathbb{B}(K)$ and we write

$$\xi \cdot x := \rho(x^{\mathrm{op}})\xi, \quad x \in M, \xi \in K.$$

We always assume that the underlying *-homomorphisms for left/right actions are faithful. Given another von Neumann algebra N, an *M-N bimodule* is a Hilbert space H which is both a left M-module and a right N-module structure in such a way that two actions commute:

$$x \cdot (\xi \cdot y) = (x \cdot \xi) \cdot y, \quad x \in M, y \in N, \xi \in H.$$

Example 2.13. $L^2(M)$ is a standard example of an M-M bimodule.

Given left M-modules H_1, H_2, we say a bouded linear map $T \in \mathbb{B}(H_1, H_2)$ is called *M-linear*, if T commutes with M-actions:

$$T(a \cdot \xi) = a \cdot (T\xi), \quad a \in M,\ \xi \in H_1.$$

Denote by ${}_M\mathbb{B}(H_1, H_2)$ the space of all bounded M-linear maps from H_1 to H_2. We say that H_1 and H_2 are isomorphic as a left M-module, if there is a unitary $u \in {}_M\mathbb{B}(H_1, H_2)$. We also use the notation ${}_M\mathbb{B}(H_1) := {}_M\mathbb{B}(H_1, H_1)$.

Lemma 2.14. *Two left M-modules H_1 and H_2 are isomorphic, if and only if there is an invertible element $T \in {}_M\mathbb{B}(H_1, H_2)$.*

Proof. Let $T = u|T|$ be the polar decomposition of T. That is, $|T| = (T^*T)^{\frac{1}{2}} \in {}_M\mathbb{B}(H_1)$ is a positive self-adjoint operator and $u \in {}_M\mathbb{B}(H_1, H_2)$ is a partial isometry which maps $\mathrm{Ker}(T)^{\perp} \subset H_1$ onto $\overline{\mathrm{Ran}}(T) \subset H_2$. Therefore u is a unitary if and only if T is invertible. The conclusion follows from this. ☐

Definition 2.15. Let H be a left M-module. A *M-submodule* is a closed subspace K that is stable under the left M-action.

Theorem 2.16. *Let M be a II_1 factor with the faithful normal tracial state τ, and let H be a separable left M-module. Then H is isomorphic to a M-submodule of $L^2(M) \otimes \ell^2(\mathbb{N})$. Here, the left M-action on $L^2(M)^{\oplus\infty} (\cong L^2(M) \otimes \ell^2(\mathbb{N}))$ is given by*

$$a \cdot (\xi_n)_{n=1}^{\infty} = (a\xi_n)_{n=1}^{\infty}, \quad a \in M,\ (\xi_n)_{n=1}^{\infty} \in L^2(M)^{\oplus}.$$

Proof. First assume that there is a cyclic vector $\xi \in H$ for the left M-action (i.e., $M\xi$ is dense in H). Then $\varphi = \omega_\xi = \langle \cdot \xi, \xi \rangle$ is a normal state on M. Then by Theorem 2.4, there exists $\xi_\varphi \in L^2(M)_+$ such that $\varphi = \omega_{\xi_\varphi}$. Then the map $V\colon M\xi \to M\xi_\varphi$ given by $V(x\xi) := x\xi_\varphi$ exntends to an isometry $V\colon H \to L^2(M)$. It is clear that V commutes with left M-actions on H and $^2(M)$. Therefore $V \in {}_M\mathbb{B}(H, L^2(M))$. Therefore by Lemma 2.14, H is isomorphic to the M-submodule $V(H)$ of $L^2(M)$.

Now let H be a general left M-module. Since we assume that H is separable, there exists (use Zorn's lemma) a coutable family $(\xi_n)_{n\in I}$ of nonzero vectors in H such that $H_i = \overline{M\xi_i}$ $(i \in I)$ satisfies

$$H_i \perp H_j \ (i \neq j), \qquad H = \bigoplus_{i\in I} H_i.$$

Clearly, each H_i is a left M-submodule of M with a cyclic vector ξ_i. Thus there exists by the first step a left M-linear isometry $V_i\colon H_i \to L^2(M)$. We then see that $V\colon H = \bigoplus_{i\in I} H_i \to L^2(M)^{\oplus I} \subset L^2(M)^{\oplus\infty}$,

$$V(\eta_i)_{i\in I} = (V_i\eta_i)_{i\in I}, \qquad (\eta_i)_{i\in I} \in H$$

defines an isomorphism of H onto $V(H) \subset L^2(M)^{\oplus\infty}$ as a left M-module. □

Let H be a left M-module. Then by Theorem 2.16, there is an M-linear isometry $V\colon H \to L^2(M)^{\oplus\infty}$. We have $VV^* \in {}_M\mathbb{B}(L^2(M)^{\oplus\infty})$. If we identify $L^2(M)^{\oplus\infty} = L^2(M) \otimes \ell^2(\mathbb{N})$, then $\mathbb{B}(L^2(M)^{\oplus\infty}) = (M'\overline{\otimes}\mathbb{B}(\ell^2))$. So the projection $p' = VV^*$ is considered as an infinite matrix $p' = [p'_{ij}]$ with entries in $p'_{ij} \in M' = JMJ$. If we set $p_{ij} = Jp'_{ij}J \in M$ $(i, j \in \mathbb{N})$, we see that as a left M-module,

$$H \cong p'(L^2(M) \otimes \ell^2(\mathbb{N})) = L^2(M) \otimes \ell^2(\mathbb{N})p.$$

Definition 2.17. Define the *von Neumann dimension* $\dim_M H$ of H as

$$\dim_M H := \sum_{n=1}^{\infty} \tau(p_{nn}) = \tau' \otimes \mathrm{Tr}(VV^*).$$

Here, τ' is the unique tracial state on the II_1 factor $JMJ = M'$ given by $\tau'(x') := \tau(Jx'J)$ $(x' \in M')$ and $\tau' \otimes \mathrm{Tr}([x'_{ij}]) = \sum_{n=1}^{\infty} \tau'(x_{nn})$ give us a (what is called, whose definition we omit) semifinite, normal and faithful tracial weight on the II_∞ factor (definition omitted, or think of the next as the definition) $M'\overline{\otimes}\mathbb{B}(\ell^2(\mathbb{N}))$.

Note that $\dim_M H$ does not depend on the choice of an M-linear isometry V. If $W\colon H \to L^2(M) \otimes \ell^2(\mathbb{N})$ is another such M-linear isometry, then since τ' is a trace,

$$\begin{aligned}\tau' \otimes \mathrm{Tr}(VV^*) &= \tau' \otimes \mathrm{Tr}(VW^*WV^*) = \tau' \otimes \mathrm{Tr}(WV^* \cdot VW^*)\\ &= \tau' \otimes \mathrm{Tr}(WW^*).\end{aligned}$$

Definition 2.18. Given an inclusion of II_1 factors $N \subset M$, the *Jones index*, denoted $[M : N]$, is the number $[M : N] = \dim_N(L^2(M))$.

Please consult [JS97] for details.

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