Seminar Notes: Simon Thomas, "A descriptive view of unitary group representations"

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Abstract

These are some supplementary notes for DST seminar.

1 Comments on Section 1

1.1 $\mathrm{Irr}(G,\mathcal{H})$ is Polish

Let G be a countable discrete group, and let $\mathcal{H}$ be a separable Hilbert space. Let $\mathrm{Rep}(G,\mathcal{H})$ be the space of all unitary representations of G on $\mathcal{H}$:

$$\mathrm{Rep}(G,\mathcal{H})=\{\pi\colon G\to\mathcal{U}(\mathcal{H});\pi \text{ is a homomorphism.}\}$$

Note that $\mathcal{U}(\mathcal{H})$ is a Polish group when endowed with SOT (strong operator topology): to see this, let $\{\xi_n\}_{n\in I}$ be a countable orthonormal basis (with $I\subset\mathbb{N}$, $I=\mathbb{N}$ if $\mathcal{H}$ is infinite-dimensional), and

$$d(u,v):=\sum_{n=1}^{\infty}\frac{1}{2^n}\{\|(u-v)\xi_n\|+\|(u^*-v^*)\xi_n\|\},\qquad u,v\in\mathcal{U}(\mathcal{H}_\infty).$$

After a not-so-short-but-routine argument, one sees that $\mathcal{U}(\mathcal{H})$ is SOT-separable, that $d$is a metric compatible with SOT, and any d-Cauchy sequence in $\mathcal{U}(\mathcal{H})$ has a d-limit in $\mathcal{U}(\mathcal{H})$.
Anyway since $\mathcal{U}(\mathcal{H})$ is Polish, so is the countable product $\prod_{g\in G}\mathcal{U}(\mathcal{H})$, and $\mathrm{Rep}(G,\mathcal{H})$ is a SOT-closed subspace of $\prod_G\mathcal{U}(\mathcal{H})$. We show that the subspace $\mathrm{Irr}(G,\mathcal{H})$ of all irreducible unitary representations of G on $\mathcal{H}$ is a G_δ subset of $\mathrm{Rep}(G,\mathcal{H})$, whence a Polish subspace:

Proposition 1.1. *$\mathrm{Irr}(G,\mathcal{H})$ is a G_δ subset of $\mathrm{Rep}(G,\mathcal{H})$.*

We show the proposition in the case $\dim(\mathcal{H})=\infty$.

Lemma 1.2. *The closed unit ball $\mathbb{B}(\mathcal{H})_1=\{x\in\mathbb{B}(\mathcal{H});\|x\|\leq 1\}$ of $\mathbb{B}(\mathcal{H})$ is SOT-separable.*

Proof. Let $\{\xi_n\}_{n=1}^\infty$ be a dense subset of $\mathcal{H}$. Then define $\Phi\colon\mathbb{B}(\mathcal{H})_1\to\prod_{n\in\mathbb{N}}\mathcal{H}$ by

$$\Phi(x):=(x\xi_n)_{n=1}^\infty,\qquad x\in\mathbb{B}(\mathcal{H})_1.$$

We show that Φ is a homeomorphism of $\mathbb{B}(\mathcal{H})_1$ onto its range. To see that Φ is injective, assume that $\Phi(x)=0$. Then $x\xi_n=0$ for every n. Then for every $\xi\in mathcalH$, there exists a sequence $(\xi_{n_k})_{k=1}^\infty$ such that $\xi_{n_k}\to\xi$. Then since x is bounded, $x\xi=\lim_{k\to\infty}x\xi_{n_k}=0$. Hence Φ is injective.
Next let $(x_i)_{i=1}^\infty$ be a net in $\mathbb{B}(\mathcal{H})_1$ and let $x\in\mathbb{B}(\mathcal{H})$. Then $x_i\stackrel{\mathrm{SOT}}{\to}x\Rightarrow\forall n\ x_i\xi_n\to x\xi_n$ is clear. Thus Φ is continuous. To see that Φ is open, assume that $\forall n\in\mathbb{N}\ [x_i\xi_n\stackrel{i\to\infty}{\to}x\xi_n]$ holds. Then for every $\xi\in\mathcal{H}$ and $\varepsilon>0$, by density of $\{\xi_n\}_{n=1}^\infty$ in $\mathcal{H}$, there exists $n\in\mathbb{N}$ such that $\|\xi_n-\xi\|<\varepsilon/3$. Then since $x_i\xi_n\to x\xi_n$, there exists $i_0\in I$ such that for every $i\geq i_0$, $\|x_i\xi_n-x\xi_n\|<\varepsilon/3$. It then follows that (use $\|x_i-x\|\leq\|x_i\|+\|x\|\leq 2$)

$$\begin{aligned}\|x_i\xi-x\xi\|&\leq\|(x_i-x)(\xi-\xi_n)\|+\|(x_i-x)\xi_n\|\\&<\|x_i-x\|\|\xi-\xi_n\|+\|(x_i-x)\xi_n\|\\&<\varepsilon.\end{aligned}$$

This shows that $x_i\xi \to x\xi$. Thus $x_i \to x$ (SOT) holds. Hence $\mathbb{B}(\mathcal{H})_1$ is a subspace of a Polish space, whence separable. ☐

Definition 1.3. Let $\mathcal{S} \subset \mathbb{B}(\mathcal{H})$. The *commutant* $\mathcal{S}'$ of $\mathcal{S}$ is the set $\mathcal{S}' := \{a \in \mathbb{B}(\mathcal{H}); ax = xa, \quad \forall x \in \mathcal{S}\}$.

Theorem 1.4 (von Neumann's double commutant theorem)**.** *Let M be a *-subagebra of $\mathbb{B}(\mathcal{H})$ containing unit $1_{\mathcal{H}}$. Then $\overline{M}^{\mathrm{SOT}} = M''(= (M')')$.*

Proof. $\overline{M}^{\mathrm{SOT}} \subset M''$: Let $x \in \overline{M}^{\mathrm{SOT}}$. Then there exists a net $(x_i)_{i\in I} \subset M$ converging to x strongly. Then for every $y' \in M'$ and $\xi \in \mathcal{H}$, we have

$$xy'\xi = \lim_{i\to\infty} x_i y'\xi = \lim_{i\to\infty} y'x_i\xi = y'x\xi.$$

Since ξ is arbitrary, $xy' = y'x$ holds. Therefore $x \in (M')'$.
$\overline{M}^{\mathrm{SOT}} \supset M''$: Let $x \in M''$. We show that x is in the strong closure of M. This, by definition of SOT, amounts to proving the following: For every $\varepsilon > 0, n \in \mathbb{N}$ and $\xi_1, \cdots, \xi_n \in \mathcal{H}$, there exists $x_0 \in M$ such that

$$\|x\xi_i - x_0\xi_i\| < \varepsilon, \qquad 1 \le i \le n. \tag{1}$$

Fix $\varepsilon > 0$ and $\xi_1, \cdots, \xi_n \in \mathcal{H}$. Let $\mathcal{H}_n = \mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (n copies) and define $\pi \colon \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H}_n)$ by

$$\pi(a)(\eta_1, \cdots, \eta_n) = (a\eta_1, \cdots, a\eta_n)\ (a \in \mathbb{B}(\mathcal{H})), \eta_1, \cdots, \eta_n \in H.$$

That is, $\pi(a) = \mathrm{diag}(a, a, \cdots, a)$ regarded as an $n \times n$ matrices with entries in $\mathbb{B}(\mathcal{H})$. It is straightforward to check that the commutant of $\pi(M)$ is the set of all $n \times n$ matrices with entries in M':

$$\pi(M)' = \left\{[x'_{ij}]; x'_{ij} \in M', 1 \le i, j \le n\right\} \subset \mathbb{B}(\mathcal{H}_n) \cong M_n(\mathbb{B}(\mathcal{H})).$$

Let $E := \overline{\pi(M)(\xi_1, \cdots, \xi_n)} \subset H_n$, and let e be the projection of H_n onto E. Then E is $\pi(M)$-invariant. Therefore $E^{\perp}$ is also $\pi(M)$-invariant. This shows that $e \in \pi(M)'$. Thus $e = [e_{ij}]$, where $e_{ij} \in M'$. It is then clear that since $x \in M''$, $\pi(x) = \mathrm{diag}(x, x, \cdots, x)$ commutes with e:

$$\pi(x)e = [xe_{ij}] = [e_{ij}x] = e\pi(x).$$

It then follows that (since $1 \in M$, $(\xi_1, \cdots, \xi_n) \in E$ holds)

$$\pi(x)(\xi_1, \cdots, \xi_n) = \pi(x)e(\xi_1, \cdots, \xi_n) = e\pi(x)(\xi_1, \cdots, \xi_n) \in E.$$

Therefore there exists $x_0 \in M$ such that

$$\|(\pi(x) - \pi(x_0))(\xi_1, \cdots, \xi_n)\|^2 = \sum_{i=1}^{n} \|x\xi_i - x_0\xi_i\|^2 < \varepsilon^2.$$

This implies (1). Therefore $x \in \overline{M}^{\mathrm{SOT}}$. ☐

Now we are almost ready to prove G_δ-ness of $\mathrm{Irr}(G, \mathcal{H})$. Let $\mathbb{C}[G]$ be the complex group ring of G, which is as a vector space over $\mathbb{C}$ the set of all formal finite sums $\sum_{g\in G} a_g g$ ($a_g \in \mathbb{C}$) and the multiplication/adjoint is defined as

$$(\sum_{g\in G} a_g g)(\sum_{g'\in G} b_{g'} g') = \sum_{g,g'\in G} a_g b'_g (gg')$$

and

$$(\sum_{g\in G} a_g g)^* = \sum_{g\in G} \overline{a_g} g^{-1}.$$

Let $\tilde{\pi} \colon \mathbb{C}[G] \to \mathbb{B}(\mathcal{H})$ be the natural extension of π defined as

$$\tilde{\pi}\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g \pi(g), \qquad a_g \in \mathbb{C}.$$

Then it is clear that $\tilde{\pi}$ is a *-homomorphism.

Lemma 1.5. *Let $\pi_n \to \pi$ in Rep$(G,\mathcal{H})$. Then for every $y \in \mathbb{C}G$, $\tilde{\pi}_n(y) \to \tilde{\pi}(y)$* (S*OT) *for every $y \in \mathbb{C}G$. Moreover, if $y = y^* \in \mathbb{C}G$, then $(\tilde{\pi}_n(y)-i)^{-1} \to (\tilde{\pi}(y)-i)^{-1}$* (SOT).

Proof. Assume that $\pi_n \to \pi$ in Rep$(G,\mathcal{H})$. That is, $\pi_n(g) \to \pi(g)$ (SOT) for every $g \in G$. Then since $\pi(g), \pi_n(g)$ are unitaries, $\pi_n(g)^* = \pi_n(g^{-1}) \to \pi(g^{-1}) = \pi(g)^*$. That is, $\pi_n(g) \to \pi(g)$ (S*OT). Therefore $\tilde{\pi}_n(y) \to \tilde{\pi}(y)$ (S*OT) for every $y \in \mathbb{C}G$. Finally, if $y = y^*$, then for $\xi \in \mathcal{H}$, then by resolvent identity and the estimate $\|(\tilde{\pi}_n(y)-i)^{-1}\| \le 1$,

$$\begin{aligned}\|(\tilde{\pi}_n(y)-i)^{-1}\xi - (\tilde{\pi}(y)-i)^{-1}\xi\| &= \|(\tilde{\pi}_n(y)-i)^{-1}(\tilde{\pi}(y)-\tilde{\pi_n}(y))(\tilde{\pi}_n(y)-i)^{-1}\xi\| \\ &\le \|(\tilde{\pi}(y)-\tilde{\pi}_n(y))(\tilde{\pi}(y)-i)^{-1}\xi\| \\ &\overset{n\to\infty}{\to} 0,\end{aligned}$$

whence $(\tilde{\pi}_n(y)-i)^{-1} \to (\tilde{\pi}(y)-i)^{-1}$ (SOT). □

We use the next result, which is a corollary of functional calculus:

Lemma 1.6. *Let A be a *-subalgebra of $\mathbb{B}(\mathcal{H})$ with unit $1_{\mathcal{H}}$. Then for $x = x^* \in \mathbb{B}(\mathcal{H})$, $x \in \overline{A}^{\rm SOT}$ if and only if $(x-i)^{-1} \in \overline{A}^{\rm SOT}$.*

Corollary 1.7. *Let $\pi \in \mathrm{Rep}(G,\mathcal{H})$. Let $\{\xi_n\}_{n=1}^{\infty}$ be a dense subset of $\mathcal{H}$, and let $\{x_n\}_{n=1}^{\infty}$ be a* SOT-*dense subset of $\mathbb{B}(\mathcal{H})_{1,\mathrm{sa}} = \{x \in \mathbb{B}(\mathcal{H}); x = x^*,\ \|x\| \le 1\}$. Then the following conditions are equivalent.*

(i) *π is irreducible.*

(ii) *$\pi(G)'' = \mathbb{B}(\mathcal{H})$.*

(iii) *$\mathrm{span}\pi(G) = \mathrm{span}\{\pi(g); g \in G\}$ is* SOT-*dense in $\mathbb{B}(\mathcal{H})$.*

(iv) *$\forall k \in \mathbb{N}\ [x_k \in \overline{\mathrm{span}}^{\rm SOT}\pi(G)]$.*

(v) *$\forall \varepsilon > 0\ \forall k \in \mathbb{N}\ \forall n \in \mathbb{N}\ \exists y = y^* \in \mathbb{C}G\ [\ \max_{1\le i\le n} \|(x_k-i)^{-1}\xi_i - (\tilde{\pi}(y)-i)^{-1}\xi_i\| < \varepsilon.]$*

Proof. (i)⇔ $\pi(G)' = \mathbb{C}1_{\mathcal{H}}$ by Schur's lemma. This implies that $\pi(G)'' = \mathbb{B}(\mathcal{H})$. Conversely, assume (ii). It is easy to see that for $\mathcal{S} \subset \mathbb{B}(\mathcal{H})$, $\mathcal{S}''' = \mathcal{S}'$. Thus $\pi(G)''' = \pi(G)' = \mathbb{B}(\mathcal{H})' = \mathbb{C}1$, whence (i). Let $A = \mathrm{span}\pi(G)$. Then A is a *-subalgebra of $\mathbb{B}(\mathcal{H})$ with unit $1_{\mathcal{H}}$, and $A'' = \pi(G)''$. Therefore by Theorem 1.4, we have $\overline{\mathrm{span}}^{\rm SOT}\pi(G) = \pi(G)''$. This shows (ii)⇔(iii).
(iii)⇒(iv) is trivial.
(iv)⇒(iii): if $x_k \in \overline{A}^{\rm SOT}$ for every $k \in \mathbb{N}$, then since $\{x_k\}_{k=1}^{\infty}$ is SOT-dense in $\mathbb{B}(\mathcal{H})_{1,\mathrm{sa}}$, $\mathbb{B}(\mathcal{H})_{\mathrm{sa}} \subset A$. Therefore $\mathbb{B}(\mathcal{H}) \subset A$, since every $x \in \mathbb{B}(\mathcal{H})$ is a sum of two self-adjoint operators $x = x_1 + ix_2, x_1 = \frac{x+x^*}{2}, x_2 = \frac{x-x^*}{2i}$.
(iv)⇒(v): Since $x_k \in \overline{A}^{\rm SOT}$, $(x_k-i)^{-1} \in \overline{A}^{\rm SOT}$ by Lemma 1.6. Then (v) is immediate from the definition of SOT.
(v)⇒(iv) This is a nontrivial part (see Remark 1.8). Since $\mathbb{B}(\mathcal{H})_{1,\mathrm{sa}}$ spans $\mathbb{B}(\mathcal{H})$ and $\{x_k\}_{k=1}^{\infty}$ is SOT-dense in $\mathbb{B}(\mathcal{H})_{1,\mathrm{sa}}$, by Lemma 1.6 it suffices to show that for every $k \in \mathbb{N}$, $(x_k-i)^{-1}$ is approximated strongly by nets of elements of the form $(\tilde{\pi}(y)-i)^{-1}$, where $y = y^* \in \mathbb{C}G$[1]. Given $k \in \mathbb{N}, \eta_1, \cdots, \eta_n \in \mathcal{H}$ and $\varepsilon > 0$. There exists $j_1, \cdots, j_n \in \mathbb{N}$ such that $\|\eta_m - \xi_{j_m}\| < \varepsilon/3$ for $m = 1, \cdots, n$. Then by (v) there exists $y = y^* \in \mathbb{C}G$ such that $\|(x_k-i)^{-1}\xi_{n_m} - (\tilde{\pi}(y)-i)^{-1}\xi_{n_m}\| < \varepsilon/3$. Then for each $m = 1, \cdots, n$ (we use the fact $\|(x_k-i)^{-1} - (\tilde{\pi}(y)-i)^{-1}\| \le 2$),

$$\begin{aligned}\|(x_k-i)^{-1}\eta_m - (\tilde{\pi}(y)-i)^{-1}\eta_m\| &\le \|\{(x_k-i)^{-1} - (\tilde{\pi}(y)-i)^{-1}\}(\eta_m - \xi_{n_m})\| \\ &\quad + \|(x_k-i)^{-1}\xi_{n_m} - (\tilde{\pi}(y)-i)^{-1}\xi_{n_m}\| \\ &< \varepsilon.\end{aligned}$$

This shows that $(x_k-i)^{-1}$ is SOT-approximated by elements of the form $(\tilde{\pi}(y)-i)^{-1}, y \in \mathbb{C}G$. Therefore (iv) holds. □

[1]If $a = \tilde{\pi}(y)$ satisfies $a = a^*$, then we may assume $y \in \mathbb{C}G$ to be self-adjoint: since $\tilde{\pi}$ is a *-homomorphism, we have $\tilde{\pi}(y) = \tilde{\pi}(y'), y' = \frac{y+y^*}{2}$.

Remark 1.8 (Some technicality)**.** The only troublesome point in the above proof is (v)⇒(iv), and this is where we use resolvent convergence: Note that we cannot prove the next (v') implies (iv):

$$(\mathrm{v}')\forall\varepsilon>0\ \forall k\in\mathbb{N}\ \forall n\in\mathbb{N}\ \exists y\in\mathbb{C}G\ [\ \max_{1\le i\le n}\|x_k\xi_i-y\xi_i\|<\varepsilon.],$$

because this only tells you that there is a net $(y_i)_{i\in I}$ in $A=\mathrm{span}\pi(G)$ such that $y_i\xi_n\to x\xi_n$ for every $n\in\mathbb{N}$. Since $\{\xi_n\}_{n=1}^{\infty}$ is dense in $\mathcal{H}$, we may want to conclude that $y_i\to x$ (SOT), but *we do not have control over* $\sup_{i\in I}\|y_i\|$! Therefore there could exist $\xi\in\mathcal{H}$ such that $y_i\xi\to x\xi$ does *not* hold.
We might come up with a use of Kaplansky's density: any $x\in\overline{A}^{\mathrm{SOT}}$ is approximated by a net $(y_i)_{i\in I}\in A$ with the norm bound $\sup_{i\in I}\|y_i\|\le\|x\|$, such that $y_i\to x$ (SOT). This might sound like a solution, but *not quite*. In fact, if we use this Kaplansky it is easy to prove that (v") and (iv) are equivalent:

$$(\mathrm{v}'')\ \forall\varepsilon>0\ \forall k\in\mathbb{N}\ \forall n\in\mathbb{N}\ \exists y\in\mathbb{C}G\ [\ \|\tilde{\pi}(y)\|\le 1\ \&\ \max_{1\le i\le n}\|x_k\xi_i-y\xi_i\|<\varepsilon.]$$

This is mathematically correct, but then the set $\{\pi\in\mathrm{Rep}(G,\mathcal{H});\|\tilde{\pi}(y)\|\le 1\ \&\ \max_{1\le i\le n}\|x_k\xi_i-y\xi_i\|<\varepsilon\}$ is *not open*, which is a serious problem (of course $\{\pi;\|\tilde{\pi}(y)\|\le 1\}$ is closed by lower-SOT-semicontinuity of the operator norm, but $\{\pi;\|\tilde{\pi}(y)\|<r\}$ is not open) for proving G_δ-ness of $\mathrm{Irr}(G,\mathcal{H})$. On the other hand, the resolvent of a self-adjoint operator, $(\tilde{\pi}(y_i)-i)^{-1}$ is always uniformly norm-bounded (by 1), regardless of whether $\{\tilde{\pi}(y_i)\}$ is actually uniformly bounded or not. That's why we use resolvents.

Proof of Proposition 1.1. By Corollary 1.7(i)⇔(v), we have

$$\mathrm{Irr}(G,\mathcal{H})=\bigcap_{\varepsilon\in\mathbb{Q}_{>0}}\bigcap_{k,n,\in\mathbb{N}}\bigcup_{y=y^*\in\mathbb{C}[G]}\bigcap_{i=1}^{n}\{\pi\in\mathrm{Rep}(G,\mathcal{H});\ \|(x_k-i)^{-1}\xi_i-(\tilde{\pi}(y)-i)^{-1}\xi_i\|<\varepsilon\}$$

By Lemma 1.5, the set

$$\bigcap_{i=1}^{n}\{\pi\in\mathrm{Rep}(G,\mathcal{H});\ \|(x_k-i)^{-1}\xi_i-(\tilde{\pi}(y)-i)^{-1}\xi_i\|<\varepsilon\}$$

is open. This shows that $\mathrm{Irr}(G,\mathcal{H})$ is G_δ in $\mathrm{Rep}(G,\mathcal{H})$. □

1.2 Finite-dimensional representations are irrelevant

Theorem 1.9 (Mackey)**.** *Let $\alpha\colon G\curvearrowright X$ be a continuous action of a compact Polish group G on a Polish space X. Then the associated orbit equivalence relation E_G^X is smooth.*

Proof. Let dg be the normalized Haar measure on G. We denote by $[x]_G=G\cdot x$ the G-orbit of $x\in X$. Let d be a compatible complete metric on X with diameter 1. Define $d_1\colon X\times X\to[0,1]$ by

$$d_1(x,y):=\int_G d(gx,gy)dg,\qquad x,y\in X.$$

Since $\sup_{x,y}d(x,y)=1$, d_1 is well-defined.

Step 1. d_1 is a metric on X compatible with the topology. Moreover, d_1 is G-invariant: $d_1(gx,gy)=d_1(x,y)$ $(x,y\in X,g\in G)$.
For $x,y\in X,g\in G$, we have (by the right-invariance of dg)

$$d_1(gx,gy)=\int_G d(hgx,hgy)dh=\int_G d(hx,hy)d(hg^{-1})=\int_G d(hx,hy)dh=d_1(x,y).$$

Positivity and triangle inequality are easy to check. If $d_1(x,y)=0$, then $d(gx,gy)=0$ for almost every g. In particular, there exists $g\in G$ for which $d(gx,gy)=0$. This shows that $gx=gy$, whence

$x = y$. Let $x_n, x \in X (n \in \mathbb{N})$. Assume first that $d(x_n, x) \to 0$. Then $d(gx_n, gx) \to 0$ $(g \in G)$. Then since $d(gx_n, gx) \le 1$, Lebesgue convergence theorem shows that

$$\lim_{n\to\infty} d_1(x_n, x) = \int_G \lim_{n\to\infty} d(gx_n, gx)dg = 0.$$

Conversely, assume that $d_1(x_n, x) \to 0$, and we show that $d(x_n, x) \to 0$. Assume by contradiction that this is not the case. By passing to a subsequence, we may assume that $\inf_n d(x_n, x) = c > 0$. Since the map $f_n \colon G \ni g \mapsto d(gx_n, gx) \in [0, 1]$ is continuous and G is compact, there exists $g_n \in G$ such that $d(g_n x_n, g_n x) = \min_{g\in G} d(gx_n, gx)$. Moreover, by compactness of G again, by passing to a further subsequence, we may assume that g_n converges to some $g \in G$. Then

$$m_n = \int_G m_n dg \le \int_G d(gx_n, gx)dg \overset{n\to\infty}{\to} 0.$$

Therefore $d(g_n x_n, g_n x) \to 0$. Then for $n, m \in \mathbb{N}$ we have

$$\begin{aligned} d(g_n x_n, g_m x_m) &\le d(g_n x_n, g_n x) + d(g_n x, g_m x) + d(g_m x, g_m x_m) \\ &\overset{n,m\to\infty}{\to} 0 + d(gx, gx) + 0 = 0. \end{aligned}$$

Therefore $\{g_n x_n\}_{n=1}^{\infty}$ is d-Cauchy. Since d is complete, it has a d-limit $x' \in X$. Thus

$$d(x', gx) = \lim_{n\to\infty} d(g_n x_n, g_n x) = 0,$$

whence $x' = gx$. This shows, by the continuity of $G \times X \ni (g, x) \mapsto gx \in X$, that

$$x_n = g_n^{-1}(g_n x_n) \overset{n\to\infty}{\to} g^{-1}gx = x.$$

which contradicts $\inf_n d(x_n, x) > 0$. Therefore d_1 is a compatible metric.

Step 2. There exists a countable family $\{E_{n,m}; n, m \in \mathbb{N}\}$ of Borel subsets of X which separates E_G^X, in the sense that if $[x]_G \ne [y]_G$, then there exists (n, m) such that $1_{E_{n,m}}(x) \ne 1_{E_{n,m}}(y)$.

By Step 1, there exists a compatible metric d which is G-invariant: $d(gx, gy) = d(x, y)$ $(g \in G, x, y \in X)$. We fix such d. Let $\{x_n\}_{n=1}^{\infty}$ be a countable dense subset of X. Since $G \ni g \mapsto gx \in X$ is continuous for every $x \in X$, the compactness of G shows that each $[x]_G$ is a compact subset of X. Therefore if $[x]_G \ne [y]_G$ for $x, y \in X$, then the distance between the orbits is strictly positive:

$$d([x]_G, [y]_G) = \inf\{d(gx, hy); g, h \in G\} > 0.$$

Moreover, since d is G-invariant and G is compact,

$$d([x]_G, [y]_G) = \min_{h\in G} d(x, hy) = \min_{g\in G} d(gx, y).$$

Therefore $d([x]_G, [y]_G) = d(x, [y]_G) = d([x]_G, y)$. Now define

$$E_{n,m} := \{x \in X; d([x_m]_G, [x]_G) = d([x_m]_G, x) < \tfrac{1}{n}\}, \quad n, m \in \mathbb{N}.$$

Then each $E_{n,m}$ is open hence Borel. We show that $\{E_{n,m}\}$ separates points in E_G^X. Let $x, y \in X$ be such that $[x]_G \ne [y]_G$. Then $r = d([x]_G, [y]_G) > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{r}{3}$. Since $\{x_n\}$ is dense, there exists $m \in \mathbb{N}$ such that $d(x, x_m) < \frac{1}{n}$, so that $x \in E_{n,m}$. Then for every $g \in G$,

$$\begin{aligned} d(gx_m, y) &\ge d(y, gx) - d(gx, gx_m) \\ &\ge d(y, [x]_G) - d(x, x_m) \\ &\ge r - \frac{1}{n} \ge \frac{2}{n}. \end{aligned}$$

Therefore $d([x_m]_G, y) \geq \frac{2}{n} > \frac{1}{n}$. This shows that $y \notin E_{n,m}$.

Step 3. There exists a Borel map $f\colon X \to 2^\omega$ satisfying $xE_G^X y \Leftrightarrow f(x) = f(y)$. That is, E_G^X is smooth.

This last step is explained in any standard textbook. Fix a bijection $\langle \cdot\, , \, \cdot \rangle\colon \omega^2 \to \omega$. Then define $F_{\langle n,m\rangle} := E_{n,m}$ $(n, m \in \mathbb{N})$, and $f(x) = (1_{F_n}(x))_{n\in\omega}$. Then by Step 2, we have $xE_G^X y \Leftrightarrow f(x) = f(y)$. It is clear that f is Borel. □

Corollary 1.10. *If $\mathcal{H}$ is a finite-dimensional Hilbert space, then the unitary equivalence relation $\approx_G$ on $\mathrm{Irr}(G, \mathcal{H})$ is smooth.*

Proof. If $\dim(\mathcal{H}) < \infty$, then $\mathcal{U}(\mathcal{H})$ is a compact Polish group. And $\cong_G$ is an orbit equivalence realation of a continuous $\mathcal{U}(\mathcal{H})$ by conjugacy. Therefore the conclusion holds by Mackey's Theorem 1.9. □

2 §3 The representation universality of $\mathbb{F}_2$

2.1 Induced representations

In §3, induced representations play crucial roles. We therefore discuss some properties of them. Let G be a countable discrete group and H a subgroup of G. Assume that we are given a unitary representation π of H on the Hilbert space $\mathcal{H}_\pi$. Let $q\colon G \to G/H$ be the canonical quotient map. We let $\mathcal{K}_\pi^0$ be the space of all functions $f\colon G \to \mathcal{H}_\pi$ such that

(i) $q(\mathrm{supp}(f)) \subset G/H$ is finite.

(ii) $f(gh) = \pi(h)^{-1} f(g), \quad g \in G, h \in H$.

Note that (i) is much weaker than "$\mathrm{supp}(f)$ is finite". Note also that if $f_1, f_2 \in \mathcal{K}_\pi^0$, then for each $g \in G, h \in H$,

$$\langle f_1(gh), f_2(gh)\rangle_{\mathcal{H}_\pi} = \langle \pi(h)^{-1} f_1(g), \pi(h)^{-1} f_2(g)\rangle_{\mathcal{H}_\pi} = \langle f_1(g), f_2(g)\rangle_{\mathcal{H}_\pi}.$$

Therefore $G \ni g \mapsto \langle f_1(g), f_2(g)\rangle$ depends only on the left coset gH. Since $q(\mathrm{supp}(f_i))$ $(i = 1, 2)$ are finite, it makes sense to define

$$\langle f_1, f_2\rangle := \sum_{gH\in G/H} \langle f_1(g), f_2(g)\rangle, \qquad f_1, f_2 \in \mathcal{K}_\pi^0.$$

Clearly $\langle\cdot,\cdot\rangle$ is a positive and sesqui-linear form on $\mathcal{K}_\pi^0$. Moreover, if $\langle f, f\rangle = 0$ for $f \in \mathcal{K}_\pi^0$, then for each $g \in G$,

$$\|f(g)\|^2 \leq \langle f, f\rangle = 0,$$

whence $f = 0$. Therefore it defines an inner-product on $\mathcal{K}_\pi^0$. Let $\mathcal{K}_\pi$ be the completion of $\mathcal{K}_\pi^0$ with respect to $\langle\cdot,\cdot\rangle$. Define $\mathrm{ind}_H^G \pi(g)$ $(g \in G)$ acting on $\mathcal{K}_\pi^0$ by

$$[\mathrm{ind}_H^G \pi(g) f](x) = f(g^{-1}x), \quad x \in G.$$

Then for each $f \in \mathcal{K}_\pi^0$ and $g \in G$,

$$\|\mathrm{ind}_H^G \pi(g) f\|^2 = \sum_{xH\in G/H} \|f(g^{-1}x)\|^2 = \sum_{xH\in G/H} \|f(x)\|^2 = \|f\|^2.$$

Therefore $\mathrm{ind}_H^G \pi(g)$ extends to an isometry on $\mathcal{K}_\pi$ and it is easily checked that it actually defines a unitary representation $\mathrm{ind}_H^G(\pi)\colon G \to \mathcal{U}(\mathcal{K}_\pi)$.

Definition 2.1. $\mathrm{ind}_H^G \pi$ is called the *induced representation* of π.

Lemma 2.2 (Bekka-de la Harpe-Valette book, Appendix E.2.1-2.3)**.** *Let H be a subgroup of a countable group G, and let $\pi_1, \pi_2 \in \mathrm{Rep}(H, \mathcal{H})$. Then*

(i) *If $\pi_1 \cong \pi_2$, then $\mathrm{ind}_H^G(\pi_1) \cong \mathrm{ind}_H^G(\pi_2)$.*

(ii) *$\mathrm{ind}_H^G(\pi_1 \oplus \pi_2) \cong \mathrm{ind}_H^G(\pi_1) \oplus \mathrm{ind}_H^G(\pi_2)$. In particular, if $\mathrm{ind}_H^G(\pi)$ is irreducible, then π is irreducible.*

Lemma 2.3. *Let $f\colon G \to \mathcal{H}_\pi$ be a finitely supported function. Then the function $\xi_f\colon G \to \mathcal{H}_\pi$ given by*

$$\xi_f(x) = \sum_{h \in H} \pi(h) f(xh)$$

belongs to $\mathcal{K}_\pi^0$. Any element in $\mathcal{K}_\pi^0$ is of this form.

Proof. We denote by $C_c(G, \mathcal{H}_\pi)$ the space of all finitely supported functions from G to $\mathcal{H}_\pi$. Let $f \in C_c(G, \mathcal{H}_\pi)$. We first see that $q(\mathrm{supp}(\xi_f))$ is finite. Let $\{g_k; k \in I\}$ be a left coset representative of H in G, with $I \subset \mathbb{N}$. So $G = \bigsqcup_{k \in I} g_k H$. Write $\mathrm{supp}(f) = \{g_{n_i} h_i^{(j)}; 1 \le i \le k, 1 \le j \le \ell_i\}$, where $n_1, n_2, \cdots, n_k$ are distinct members of I. Then $f(gh)$ $(g \in G, h \in H)$ is nonzero only if $gh = g_{n_i} h_i^{(j)}$ for some $1 \le i \le k$ and $1 \le j \le \ell_i$. Therefore

$$\xi_f(g) = \sum_{h \in H} \pi(h) f(gh) = \begin{cases} \sum_{j=1}^{\ell_i} \pi(g_{n_i}^{-1} g) f(g_{n_i} h_i^{(j)}) & (g \in g_{n_i} H,\ 1 \le i \le k) \\ 0 & (\text{otherwise}). \end{cases}$$

Therefore $q(\mathrm{supp}(\xi_f)) \subset \{g_{n_i} H; 1 \le i \le k\} = q(\mathrm{supp}(f))$, which is finite. Also, for $g \in G, h \in H$,

$$\xi_f(gh) = \sum_{h' \in H} \pi(h') f(ghh') = \sum_{h'' \in H} \pi(h^{-1} h'') f(gh'') = \pi(h)^{-1} \xi_f(g).$$

Therefore $\xi_f \in \mathcal{K}_\pi^0$. Conversely, let $\xi \in \mathcal{K}_\pi^0$. Then there exist $g_1, \cdots, g_n \in G$ such that $q(\mathrm{supp}(\xi)) = \{g_1 H, \cdots, g_n H\}$. Define $\psi \in G \to \mathbb{C}$ by $\psi(g) = 1$ if $g = g_i$ $(1 \le i \le n)$ and $\psi(g) = 0$ otherwise. Set $\varphi \in C_c(G, \mathcal{H}_\pi)$ by $\varphi(g) = \psi(g)\xi(g), g \in G$. Let $x \in G$. Then

$$\begin{aligned} \xi_\varphi(x) &= \sum_{h \in H} \pi(h) \varphi(xh) = \sum_{h \in H} \psi(xh) \pi(h) \xi(xh) \\ &= \sum_{h \in H} \psi(xh) \pi(h) \pi(h)^{-1} \xi(x) = \sum_{h \in H} \psi(xh) \xi(x). \end{aligned}$$

If $x \notin \mathrm{supp}(\xi)$, then $\xi_\varphi(x) = 0 = \xi(x)$. If $x \in \mathrm{supp}(\xi)$, then $q(x) \in \mathrm{supp}(\xi)$, whence there exists $1 \le i \le n$ and $h_i \in H$ such that $x = g_i h_i$. Then

$$\sum_{h \in H} \psi(xh) = \sum_{h \in H} \psi(g_i h_i h) = \psi(g_i) = 1.$$

Therefore $\xi_\varphi(x) = \xi(x)$ for all $x \in G$, which finishes the proof. □

Next result is important in our analysis. Recall that $\mathcal{H}_\infty$ is a fixed separable and infinite-dimensional Hilbert space.

Proposition 2.4. *Let G be a countable discrete group, and let $H \le G$ be a normal subgroup. Ret $\{g_i; i \in I\}$ be a coset representative of H. Let $\pi \in \mathrm{Rep}(H, \mathcal{H}_\infty)$ and $\theta_\pi = \mathrm{ind}_H^G \pi \in \mathrm{Rep}(G, \mathcal{K}_\pi)$.*

(i) *The following unitary equivalence (as a representation of H) holds:*

$$\theta_\pi|_H \cong \bigoplus_{k \in I} \pi^{g_k^{-1}}.$$

Here, $\pi^{g_k}(x) := \pi(g_k x g_k^{-1})$ for $x \in H$.

(ii) *Assume that π is irreducible. Then θ_π is irreducible if and only if $\pi^g \not\cong \pi$ for all $g \in G \setminus H$.*

(iii) *For each $\pi \in \mathrm{Rep}(H, \mathcal{H}_\infty)$, there exists a unitary $V_\pi \colon \mathcal{K}_\pi \to \mathcal{H}_\infty^{\oplus I} := \bigoplus_{k\in I} \mathcal{H}_\infty$ such that $\Theta_\pi \colon \mathrm{Rep}(H, \mathcal{H}_\infty) \ni \pi \mapsto V_\pi \theta_\pi V_\pi^{-1} \in \mathrm{Rep}(G, \mathcal{H}_\infty^{\oplus I})$ is Borel.*

Proof. (i) For $v \in \mathcal{H}_\pi$, we let $v_g \in C_c(G, \mathcal{H}_\pi)$ by $v_g(x) = \delta_{g,x} v$ for $x \in G$. Then $C_c(G, \mathcal{H}_\pi)$ is spanned by $\{v_g; g \in G, v \in \mathcal{H}_\pi\}$. Therefore by Lemma 2.3, $\mathcal{K}_\pi^0$ is spanned by $\{\xi_{v_g}; g \in G, v \in \mathcal{H}_\pi\}$. We denote ξ_{v_g} as $\xi_{g,v}$ $(v \in \mathcal{H}_\pi, g \in G)$. Let us compute $\xi_{g,v}$ more explicitly: for $x \in G$,

$$\xi_{g,v}(x) = \sum_{h\in H} \pi(h) v_g(xh) = \sum_{h\in H} \delta_{g,xh} \pi(h) v = \begin{cases} \pi(x^{-1}g)v & (x \in gH) \\ 0 & (x \notin gH) \end{cases}. \tag{2}$$

Also, note that

$$\xi_{gh,v} = \xi_{g,\pi(h)v}, \qquad g \in G, h \in H, v \in \mathcal{H}_\infty \tag{3}$$

To see this, let $x \in G$. Then

$$\begin{aligned} \xi_{gh,v}(x) &= \begin{cases} \pi(x^{-1}gh)v & (x \in ghH = gH) \\ 0 & (x \notin gH) \end{cases} \\ &= \begin{cases} \pi(x^{-1}g)(\pi(h)v) & (x \in gH) \\ 0 & (x \notin gH) \end{cases} \\ &= \xi_{g,\pi(h)v}(x). \end{aligned}$$

Therefore since $\{g_k; k \in I\}$ is a coset representative of H, we see that

$$\mathcal{K}_\pi^0 = \mathrm{span}\{\xi_{g_k,v}; k \in I, v \in \mathcal{H}_\infty\} \tag{4}$$

Now let $k, l \in I$, $v, v' \in \mathcal{H}_\infty$ and compute $\langle \xi_{g_k,v}, \xi_{g_l,v'} \rangle$: note that $\langle \xi_{g_k,v}(x), \xi_{g_l,v'}(x) \rangle$ depends only on the left coset of x, it follows that

$$\langle \xi_{g_k,v}, \xi_{g_l,v'} \rangle = \sum_{i\in I} \langle \xi_{g_k,v}(g_i), \xi_{g_l,v'}(g_i) \rangle. \tag{5}$$

By (2), $\langle \xi_{g_k,v}(g_i), \xi_{g_l,v'}(g_i) \rangle$ is nonzero only if $g_i \in g_k hH = g_k H$ and $g_i \in g_l H$, which happens if and only if $l = k = i$. Thus, if $k \neq l$ then $\langle \xi_{g_k,v}, \xi_{g_l,v'} \rangle = 0$ and if $k = l$,

$$\begin{aligned} \langle \xi_{g_k,v}, \xi_{g_k,v'} \rangle &= \langle \xi_{g_k,v}(g_k), \xi_{g_k,v'}(g_k) \rangle \\ &= \langle \pi(g_k^{-1} g_k)v, \pi(g_k^{-1} g_k)v' \rangle \\ &= \langle v, v' \rangle. \end{aligned}$$

Therefore

$$\langle \xi_{g_k,v}, \xi_{g_l,v'} \rangle = \delta_{k,l} \langle v, v' \rangle, \qquad k, l \in I, \ v, v' \in \mathcal{H}_\infty. \tag{6}$$

Then for each $k \in I$, define $U_k \colon \mathcal{H}_\infty \to \mathcal{K}_\pi$ by

$$U_k v := \xi_{g_k,v}, \qquad v \in \mathcal{H}_\infty$$

Then by (6), U_k is a unitary from $\mathcal{H}_\infty$ onto $\mathcal{K}_\pi^{(k)} = \{\xi_{g_k,v}; v \in \mathcal{H}_\infty\}$. In particular, the ranges of U_k and U_l $(k \neq l)$ are orthogonal and $\mathcal{K}_\pi = \bigoplus_{k\in I} \mathcal{K}_\pi^{(k)}$.
We now show that

(a) Each $\mathcal{K}_\pi^{(k)}$ is invariant under $\theta_\pi|_H$

(b) The restriction of $\theta_\pi|_H$ on $\mathcal{K}_\pi^{(k)}$ is unitarily equivalent to $\pi^{g_k^{-1}}$.

For (a), let $x, g \in G$ and $v \in \mathcal{H}_\infty$. Then

$$\begin{aligned}[\theta_\pi(g)\xi_{g_k,v}](x) &= \xi_{v,g_k}(g^{-1}x) \\ &= \begin{cases} \pi(x^{-1}gg_k)v & (g^{-1}x \in g_kH \Leftrightarrow x \in gg_kH) \\ 0 & (x \notin gg_kH) \end{cases}.\end{aligned}$$

This shows that

$$\theta_\pi(g)\xi_{g_k,v} = \xi_{gg_k,v}, \qquad g \in G,\ k \in I,\ v \in \mathcal{H}_\infty. \tag{7}$$

In particular, if $h \in H$, then $hg_k = g_k(g_k^{-1}hg_k) \in g_kH$, so $\theta_\pi(h)\xi_{g_k,v} = \xi_{hg_k,v} \in \mathcal{K}_\pi^{(k)}$, which shows (a). Moreover, we also see from this calculation and (3) that

$$\begin{aligned}\theta_\pi(h)U_kv = \theta_\pi(h)\xi_{g_k,v} &= \xi_{g_k(g_k^{-1}hg_k),v} \\ &= \xi_{g_k,\pi(g_k^{-1}hg_k)v} \\ &= U_k\pi(g_k^{-1}hg_k)v \\ &= U_k\pi^{g_k^{-1}}(h)v.\end{aligned}$$

This shows that

$$\theta_\pi(h)U_k = U_k\pi^{g_k^{-1}}(h) \quad \text{ on } \mathcal{H}_\infty.$$

This shows (b) and by $\mathcal{K}_\pi = \bigoplus_{k\in I}\mathcal{K}_\pi^{(k)}$, that

$$\theta_\pi|_H \cong \bigoplus_{k\in I}\pi^{g_k^{-1}}.$$

Next, we show (ii).
($\Rightarrow$) Assume by contradiction that there exists $g \in G \setminus H$ such that $\pi^g \cong \pi$. There exists $i \in I$ and $h_i \in H$ such that $h_ig_i^{-1} = g$. Then there exists $v \in \mathcal{U}(\mathcal{H})$ such that

$$v\pi(h_ig_ihg_i^{-1}h_i^{-1})v^* = (v\pi(h_i))\pi^{g_i}(h)(v\pi(h_i))^* = \pi(h), \qquad h \in H.$$

Thus if we set $u = v\pi(h) \in \mathcal{U}(\mathcal{H}_\infty)$, we have $u\pi^{g_i}(h)u^* = \pi(h)$, whence $\pi^{g_i} \cong \pi$. Then define $T_u \colon \mathcal{K}_\pi^0 \to \mathcal{K}_\pi^0$ by

$$T_u\xi_{g,v} := \xi_{gg_i^{-1},uv}, \qquad v \in \mathcal{H}_\infty, g \in G.$$

Note that we have to check that T_u is well-defined, because there is an ambiguity of the presentation $\xi_{g_kh_k,v_k} = \xi_{g_k,\pi(h_k)v_k}, (h \in H, v_k \in \mathcal{H}_\infty, k \in I)$. We see that

$$\begin{aligned}T_u\xi_{g_k,\pi(h_k)v_k} &= \xi_{g_kg_i^{-1},u\pi(h_k)v_k} = \xi_{g_kg_i^{-1},\pi^{g_i}(h_k)uv_k} \\ &= \pi_{g_kg_i^{-1}(g_ih_kg_i^{-1}),uv_k} = \xi_{g_kh_kg_i^{-1},uv_k}.\end{aligned}$$

So T_u is well-defined, and it satisfies

$$T_u\sum_{k\in I,\text{fin}}\xi_{g_kh_k,v_k} = \sum_{k\in I,\text{fin}}\xi_{g_kh_kg_i^{-1},uv_k}, \qquad h_k \in H, v_k \in \mathcal{H}_\infty.$$

It is clear that T_u is isometric with dense range. Thus T_u has a unique exntension to a unitary, still denoted T_u on $\mathcal{K}_\pi$. Then for every $g \in G$, $v = (v_k)_{k\in I} \in \mathcal{H}_\infty^{\oplus I}$, we have

$$\begin{aligned}T_u\theta_\pi(g)\sum_{k\in I}\xi_{g_k,v_k} &= T_u\sum_{k\in I}\xi_{gg_k,v_k} = \sum_{k\in I}\xi_{gg_kg_i^{-1},uv_k} \\ &= \theta_\pi(g)\sum_{k\in I}\xi_{g_kg_i^{-1},v_k} = \theta_\pi(g)T_u\sum_{k\in I}\xi_{g_k,v_k},\end{aligned}$$

whence $T_u \in \theta_\pi(G)'$. By Schur's lemma, $T_u = \lambda 1_{\mathcal{K}_\pi}$ for some $\lambda \in \mathbb{C}$. But if $v \in \mathcal{H}_\infty$ is any unit vector, then

$$\lambda\xi_{g_i,v} = T_u\xi_{g_i,v} = \xi_{1,uv}$$

is a unit vector. So $\lambda \neq 0$. But since $g_i \notin H$, $\xi_{g_i,v} \perp \xi_{1,uv}$, and

$$\lambda = \lambda\langle \xi_{g_i,v}, \xi_{g_i,v}\rangle = \langle \xi_{1,uv}, \xi_{g_i,v}\rangle = 0.$$

This is a contradiction.

($\Leftarrow$)Assume that $\pi^g \ncong \pi$ for every $g \in G \setminus H$. Note that since π is irreducible, we have $\pi^g(H)' = \pi(gHg^{-1})' = \pi(H)' = \mathbb{C}1_{\mathcal{H}_\infty}$ (Schur's lemma). Therefore π^g is irreducible for every $g \in G$. We show that θ_π is irreducible. Let $T \in \theta_\pi(G)'$. Then according to the identification $\mathcal{K}_\pi = \bigoplus_{k\in I} \mathcal{H}_\infty$ (recall that this means we identify $(v_k)_{k\in I}$ with $\sum_{k\in I} \xi_{g_k,v_k}$), we may view T as a matrix $[T_{ij}]_{i,j\in I}$ with elements $T_{i,j}$ in $\mathbb{B}(\mathcal{H}_\infty)$. That is, if $\boldsymbol{v} = (v_k)_{k\in I} \in \mathcal{K}_\pi$, then $T\boldsymbol{v} = \boldsymbol{w} = (w_k)_{k\in I}$, where $T_{ij} \in \mathbb{B}(\mathcal{H}_\infty)$ satisfies

$$w_k = (T\boldsymbol{v})_k = \sum_{i\in I} T_{ki} v_i.$$

Also, since $\{g_k; k \in I\}$ is a coset representative of H, for each $i, j, k \in I$, there exist unique $m = m(i,k), n = n(i,j) \in I$ and unique $\tilde{h} = \tilde{h}(i,k), h = h(i,j)$ such that

$$g_i g_{m(i,k)} = g_k \tilde{h}(i,k), \quad g_i g_j = g_{n(i,j)} h(i,j).$$

From this, it holds that (we use (3))

$$\theta(g_i)\xi_{g_j,v_j} = \xi_{g_{n(i,j)}h(i,j),v_j} = \xi_{g_{n(i,j)},\pi(h(i,j))v_j}.$$

Therefore with respect to the identification $\mathcal{K}_\pi = \bigoplus_{k\in I} \mathcal{H}_\infty$, we may view $\theta(g_i)$ $(i \in I)$ as an operator-valued matrix $[\theta(g_i)_{kj}]_{k,j\in I}$ as follows:

$$\theta(g_i)_{kj} = \delta_{n(i,j),k}\pi(h(i,j)) = \begin{cases} \pi(h(i,j)) & (k = n(i,j)) \\ 0 & (k \neq n(i,j)) \end{cases} \tag{8}$$

Also, from (i) we see that

$$\theta(h) = [\delta_{i,j}\pi^{g_i^{-1}}(h)]_{i,j\in I}], \qquad h \in H. \tag{9}$$

Since T commutes with every $\theta(h)$ $(h \in H)$, we have

$$T\theta(h) = [T_{ij}\pi^{g_j^{-1}}(h)]_{i,j} = \theta(h)T = [\pi^{g_i^{-1}} T_{ij}]_{i,j}.$$

Therefore

$$T_{ij} \in \mathrm{Hom}_H(\pi^{g_j^{-1}}, \pi^{g_i^{-1}}), \qquad i, j \in I. \tag{10}$$

Now we observe that:

Claim. For $g, g' \in G$, $\pi^g \cong \pi^{g'} \Leftrightarrow g'^{-1}g \in H$. In particular, $\pi^{g_i^{-1}} \ncong \pi^{g_j^{-1}}$ if $i, j \in I$, $i \neq j$.
To see this, assume that $\pi^g \cong \pi^{g'}$. Then there exists $u \in \mathcal{U}(\mathcal{H}_\infty)$ such that

$$u\pi(ghg^{-1})u^* = \pi(g'h(g')^{-1}), \qquad h \in H.$$

Then since $(g')^{-1}H'g' = H$, we have

$$u\pi(g\{(g')^{-1}hg'\}g^{-1})u^* = \pi(g'\{(g')^{-1}hg'\}(g')^{-1}) = \pi(h), \qquad h \in H.$$

This shows that $\pi^{g(g')^{-1}} \cong \pi$. By assumption, this shows that $g(g')^{-1} \in H$, whence the claim is proved.

By the Claim, (10), and Schur's lemma, we see that there exists $\lambda_i \in \mathbb{C}$ $(i \in I)$ such that

$$T_{ij} = \begin{cases} \lambda_i 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{11}$$

Our last task is to show that $\lambda_i = \lambda_j$ $(i, j \in I)$. But since T also commutes with each $\theta(g_i)$ $(i \in I)$. We have

$$\begin{aligned}
[T\theta(g_i)]_{kj} &= \sum_{p\in I} T_{kp}(\theta(g_i))_{pj} = T_{kk}\theta(g(i))_{kj} \\
&= \lambda_k \delta_{n(i,j),k}\pi(h(i,j)), \\
[\theta(g_i)T]_{kj} &= \sum_{p\in I} (\theta(g_i))_{kp}T_{pj} = (\theta(g_i))_{kj}T_{jj} \\
&= \lambda_j \delta_{n(i,j),k}\pi(h(i,j))
\end{aligned}$$

Since $\pi(h(i,j))$ is unitary, this shows that $\lambda_j = \lambda_k$, whence $T \in \mathbb{C}1_{\mathcal{K}_\pi}$. Therefore by Schur's lemma, $\theta_\pi = \mathrm{ind}_H^G(\pi)$ is irreducible.

Finally, we show (iii). Define $V_\pi \colon \mathcal{K}_\pi = \bigoplus_{k\in I} \mathcal{K}_\pi^{(k)} \to \mathcal{H}_\infty^{\oplus I} = \bigoplus_{k\in I} \mathcal{H}_\infty$ by

$$V_\pi(\xi_{g_k,v_k})_{k\in I} := (v_k)_{k\in I}, \quad (\xi_{g_k,v_k})_{k\in I} \in \mathcal{K}_\pi.$$

By (6), V_π is a well-defined unitary. Let $\Theta_\pi = V_\pi\theta_\pi V_\pi^{-1} \in \mathrm{Rep}(G, \mathcal{H}_\infty^{\oplus I})$ be the representation given by

$$\Theta_\pi(g) = V_\pi\theta_\pi(g)V_\pi^{-1}, \qquad g \in G.$$

We show that $\pi \mapsto \Theta_\pi$ is continuous (hence Borel). To distinguish vectors belonging to different Hilbert spaces $\mathcal{K}_\pi$ for different π we denote $\xi^\pi_{g_k,v} \in \mathcal{K}_\pi$ instead of $\xi_{g_k,v}$, making the dependence on π explicit. Assume that a sequence $(\pi_n)_{n=1}^\infty$ in $\mathrm{Rep}(H, \mathcal{H}_\infty)$ converges to $\pi \in \mathrm{Rep}(H, \mathcal{H}_\infty)$. Then $\pi_n(g) \to \pi(g)$ (SOT) for each $g \in G$. We have to show that $\Theta_{\pi_n}(g)(v_k)_{k\in I} \to \Theta_\pi(g)(v_k)_{k\in I}$ for every $\boldsymbol{v} = (v_k)_{k\in I} \in \mathcal{H}_\infty^{\oplus I}$. Once again, since $\{g_k; k \in I\}$ is a coset representative of H, for each $g \in G$ there exists a unique $h = h(g,k) \in H$ and a unique $j = j(g,k) \in I$ such that $gg_j = g_kh$. Thus we have maps $h \colon G \times I \to H$ and $j \colon G \times I \to I$. Thus for each $k \in I, g \in G$,

$$\begin{aligned}
\theta_{\pi_n}(g)\xi^{\pi_n}_{g_{j(g,k)},v_{j(g,k)}} &= \xi^{\pi_n}_{gg_{j(g,k)},v_{j(g,k)}} = \xi^{\pi_n}_{g_kh(g,k),v_k} \\
&= \xi^{\pi_n}_{g_k,\pi_n(h(g,k))v_{j(g,k)}}.
\end{aligned}$$

Therefore $\mathcal{K}_{\pi_n}^{(k)}$-component of $\theta_{\pi_n}(g)(\xi_{g_k}, v_k)_{k\in I}$ is

$$\xi^{\pi_n}_{g_k,\pi_n(h(g,k))v_{j(g,k)}}.$$

It therefore holds that (since $\pi_n(h(g,k)) \overset{n\to\infty}{\to} \pi(h(g,k))$ (SOT))

$$\begin{aligned}
\Theta_{\pi_n}(g)\boldsymbol{v} &= V_{\pi_n}\theta_{\pi_n}(g)(\xi^{\pi_n}_{g_k,v_k})_{k\in I} = V_{\pi_n}(\xi^{\pi_n}_{g_k,\pi_n(h(g,k))v_{j(g,k)}})_{k\in I} \\
&= (\pi_n(h(g,k))v_{j(g,k)})_{k\in I} \\
&\overset{n\to\infty}{\to} (\pi(h(g,k))v_{j(g,k)})_{k\in I} \\
&= \Theta_\pi(g)\boldsymbol{v}.
\end{aligned}$$

Therefore $\pi \mapsto \Theta_\pi$ is continuous. □

Finally, let us look at some simple examples of induced representations. For each integer $p \geq 1$, let $\mathbb{Z}$ act on $\mathbb{Z}$ by $\alpha^p(m)n := p^m n, m, n \in \mathbb{Z}$ and take the semidirect product $G = \mathbb{Z} \rtimes_{\alpha^p} \mathbb{Z}$. Therefore G is $\mathbb{Z} \times \mathbb{Z}$ as a set, and the group law (with group unit $(0,0)$) is

$$(n,m)(n',m') = (n + p^m n', m + m'), \quad (n,m)^{-1} = (-p^{-m}n, -m), \qquad n, m \in \mathbb{Z}.$$

$G_1 = \mathbb{Z} \times \mathbb{Z}$ is abelian, so it cannot have infinite-dimensional irreducible representation. For $p \geq 2$, it is non-abelian and in fact an ICC group:

Lemma 2.5. *The following holds.*

(i) G_p $(p \geq 2)$ *is an* ICC *group. That is, for each* $g \in G_p \setminus \{(0,0)\}$*, the conjugacy class* $\{xgx^{-1}; x \in G_p\}$ *is an infinite set.*

(ii) G_p *is therefore not abelian-by-finite.*

(iii) $G_p \cong G_q$ *if and only if* $p = q$.

Proof. (i) Let $n, m, k, l \in \mathbb{Z}$ with $(k, l) \neq (0, 0)$. Then

$$(n,m)(k,l)(n,m)^{-1} = (n + p^m k, m + l)(-p^{-m}n, -m) = (n + p^m k - p^l n, l) \tag{12}$$

If $k \neq 0$, then the set $S = \{n + p^m k - p^l n; (n,m) \in \mathbb{Z}\}$ contains (setting $n = 0$) an infinite set $\{p^m k; m \in \mathbb{Z}\}$. If $l \neq 0$, then (set $m = 0$) S contains an infinite set $\{(1 - p^l)n + k; n \in \mathbb{Z}\}$. Therefore the conugacy class of $(k, l) \neq (0, 0)$ is infinite. Thus G_p is ICC.
(ii) We show that any ICC group G is not abelian-by-finite. If there exists an abelian normal subgroup $N \triangleleft G$ of an ICC group G with finite index $n = [G : N] < \infty$. Then if $N = \{1\}$, G is a finite group, a contradiction. Thus there exists $h \in N \setminus \{1\}$. But if $\{g_1, \cdots, g_n\}$ is the coset representative of N in G, then for every $x \in G$ there is $1 \leq i \leq n$ and $h_i \in N$ such that $x = g_i h_i$. Then $xhx^{-1} = g_i h_i h h_i^{-1} g_i^{-1} = g_i h g_i^{-1}$ because N is abelian. Thus the conjugacy class of h is just $\{g_i h g_i^{-1}; 1 \leq i \leq n\}$, a contradiction to ICC property. Thus G is not abelian-by-finite.
(iii) Let $2 \leq p \leq q$ be integers, and assume that there exists a group isomorphism $\Phi \colon G_p \to G_q$. Let $a = (1, 0), b = (0, 1) \in G_p$ and set $a' = f(a) = (n_1, m_1), b' = f(b) = (n_2, m_2) \in G_q$. Then

$$bab^{-1} = (0, 1)(1, 0)(0, -1) = (p, 0) = a^p.$$

So $b'a'(b')^{-1} = \Phi(bab^{-1}) = \Phi(a)^p = (a')^p$. On the other hand,

$$b'a'(b')^{-1} = (n_2, m_2)(n_1, m_1)(-q^{-m_2}n_2, -m_2) = (n_2 + q^{m_2}n_1 - q^{m_1}n_2, m_1).$$

On the other hand, the second component of $(a')^p$ is pm_2 as can be seen by induction. Thus $m_1 = pm_1$. Since $p \geq 2$, we have $m_1 = 0$. Then $a' = (n_1, 0)$, and $(a')^p = (pn_1, 0)$. Therefore

$$b'a'(b')^{-1} = (n_2 + q^{m_2}n_1 - n_2, 0) = (q^{m_2}n_1, 0) = (pn_1, 0),$$

whence $q^{m_2}n_1 = pn_1$. If $n_1 = 0$, then $a' = 0$. Therefore $f(G_p) = f(< b >)$, which is abelian. But G_q is non-abelian, whence $f(G_p) \subsetneq G_q$, a contradiction. Therefore $n_1 \neq 0$, and $q^{m_2} = p$. If $m_2 \leq 0$, then $q^{m_2} \leq 1 < p$, so $m_2 \geq 1$. But then $p \leq q \leq q^{m_2} = p$, whence $m_2 = 1$ and $p = q$. □

So G_p $(p \geq 2)$ are all "amenable non-type I groups". Thus they have infinite-dimensional irreducible representations (see Sec. 1). We can construct such a family of irreducible unitary representations by induction. Fix $p \geq 2$ and consider $H = \mathbb{Z} \times \{0\} \triangleleft G_p$. For each $0 < \alpha < 1$, define an irreducible one-dimensional unitary representation π_α of $H \cong \mathbb{Z}$ by

$$\pi_\alpha((n, 0)) = e^{2\pi i n\alpha}1, \quad n \in \mathbb{Z}.$$

Then $\{(0, k); k \in \mathbb{Z}\}$ is a coset representative of H in G_p.

Proposition 2.6. *Set* $\Theta_\alpha = \mathrm{ind}_H^{G_p}(\pi_\alpha)$*. The following holds.*

(i) *If* $\alpha \in (0, 1) \setminus \mathbb{Q}$*, then* Θ_α *is an infinite-dimensional irreducible unitary representation of* G_p.

(ii) *Let* $\alpha, \beta \in (0, 1) \setminus \mathbb{Q}$ *with* $\alpha/\beta \notin \mathbb{Q}$*. Then* $\Theta_\alpha \not\cong \Theta_\beta$.

Proof. (i) Since $\mathcal{K}_{\pi_\alpha} = \bigoplus_{\mathbb{Z}} \mathbb{C} \cong \ell^2(\mathbb{Z})$, Θ_α is an infinite-dimensional unitary representation. To show the irreducibility, by Proposition 2.4 (ii), it suffices to check that $\pi_\alpha^{(0,k)} \not\cong \pi_\alpha$ if $k \neq 0$. For $n \in \mathbb{Z}$,

$$\begin{aligned}\pi_\alpha^{(0,k)}((n, 0)) &= \pi_\alpha((0, k)(n, 0)(0, k)^{-1}) = \pi_\alpha((p^k n, k)(0, -k)) \\ &= \pi_\alpha(p^k n, 0) = e^{2\pi i (p^k n)\alpha} \\ &\neq e^{2\pi i n\alpha} = \pi_\alpha((n, 0)),\end{aligned}$$

because α is irrational and $(p^k-1)\alpha \notin \mathbb{Q}$. Thus $\pi_\alpha^{(0,k)} \ncong \pi_\alpha$ if $k \neq 0$. Therefore Θ_α is irreducible.
(ii) Assume by contradiction that $\Theta_\alpha \cong \Theta_\beta$ for irrational $0 < \alpha < \beta$ with $\alpha/\beta \notin \mathbb{Q}$. Then by Proposition 2.4, one has

$$\Theta_\alpha|_H = \bigoplus_{k\in\mathbb{Z}} \pi_\alpha^{(0,k)} \cong \bigoplus_{k\in\mathbb{Z}} \pi_\beta^{(0,k)} = \Theta_\beta|_H.$$

Then since $\pi_\alpha^{(0,k)}, \pi_\beta^{(0,l)}$ $(k,l\in\mathbb{Z})$ are irreducible, as in the proof of Theorem 2.12 below, there exists $k\in\mathbb{Z}$ such that $\pi_\alpha \cong \pi_\beta^{(0,k)}$. Then for each $n\in\mathbb{Z}$, $e^{2\pi i n\alpha} = e^{2\pi i p^k n\beta}$. Then $\alpha - p^k\beta \in \mathbb{Z}$, which is a contradiction. □

2.2 $\mathbb{F}_2$ is representation universal

Let $\mathbb{F}_2 = \langle a, b\rangle$ be the free group on two generators a, b, let $h\colon \mathbb{F}_2 \to \mathbb{Z}$ be the homomorphism with $h(a) = 1, h(b) = 0$. Let $H = \mathrm{Ker}(h) \triangleleft \mathbb{F}_2$. H is a subgroup of $\mathbb{F}_2$, whence free by Schreier's Theorem.

Lemma 2.7. *The following holds.*

1. $T = \{a^n; n\in\mathbb{Z}\}$ *is a coset representative of* H *in* $\mathbb{F}_2$.
2. $H = \langle a^n b a^{-n}, n\in\mathbb{Z}\rangle \cong \mathbb{F}_\infty$.

Theorem 2.8 (Explicit form of Schreier's Theorem (Serre Book))**.** *Let* G *be a free group with basis* S*, and let* H *be a subgroup of* G*.*

(a) *One can choose a set* T *of representatives of* $H\backslash G$ *satisfying the following condition:*
(*) If $t\in T$ *has the reduced decomposition*

$$t = s_1^{\varepsilon_1}\cdots s_n^{\varepsilon_n}\ (s_i\in S, \varepsilon_i = \pm 1,\ \varepsilon_i = \varepsilon_{i+1} \text{ if } s_i = s_{i+1}),$$

then all the partial products $1, s_1^{\varepsilon_1}, s_1^{\varepsilon_1}s_2^{\varepsilon_2}, \cdots, s_1^{\varepsilon_1}\cdots s_n^{\varepsilon_n}$ *belong to* T.

(b) *Let* T *be as above and let* $W = \{(t,s)\in T\times S, ts\notin T\}$*. If* $(t,s)\in W$*, set* $h_{t,s} = tsu^{-1}$ *where* $u\in T$ *is such that* $Hts = Hu$*. Then*

$$R = \{h_{t,s}; (t,s)\in W\}$$

is a basis for H*.*

Proof of Lemma 2.7. (1) Given $g\in\mathbb{F}_2$, and let $n = h(g)\in\mathbb{Z}$. Then $h(ga^{-n}) = h(g) - n = 0$, whence $ga^{-n}\in H \Leftrightarrow g\in Ha^n$. This shows that $\mathbb{F}_2 = \bigcup_{n\in\mathbb{Z}} Ha^n$. Moreover, if $n, m\in\mathbb{Z}$ with $n\neq m$, then $h(a^n) = n \neq m = h(a^m)$. This shows that $Ha^n\cap Ha^m = \emptyset$. Therefore $\mathbb{F}_2 = \bigsqcup_{n\in\mathbb{Z}} Ha^n$ gives the right coset decomposition of H.
(2) $T = \{a^n; n\in\mathbb{Z}\}, S = \{a, b\}$ satisfies the condition (*) in Theorem 2.8. In this case, $W = \{(t,s)\in T\times S; ts\notin T\} = \{(a^n, b); n\in\mathbb{Z}\}$. Given $(t,s) = (a^n, b)\in W$, find $u = a^l\in T$ such that $Ha^n b = Ha^l$. This condition means $a^n b a^{-l}\in H$, whence $0 = h(a^n b^{-l}) = n - l$, whence $l = n$, and $h_{t,s} = tsu^{-1} = a^n b^{-n}$. Thus by Theorem 2.8, $\{a^n b a^{-n}; n\in\mathbb{Z}\}$ is a basis of H. □

Define $c_n := b_{2^n} (n\in\mathbb{N}), b_n := a^n b a^{-n}$ and $C = \{c_n; n\in\mathbb{N}\}$. Since $S = \{a^n b a^{-n}; n\in\mathbb{Z}\}$ is a free generating set of H, the subgroup $< C >$ generated by C is isomorphic to $\mathbb{F}_\infty$. Thus we identify $\mathbb{F}_\infty =< C >$, which is a subgroup of H. Then for each $\pi\in\mathrm{Rep}(\mathbb{F}_\infty)$ we set $\tilde{\pi}\in\mathrm{Rep}(H)$ as the unique representation satisfying

$$\tilde{\pi}(s) = \begin{cases}\pi(s) & (s\in C)\\ 1 & (s\in S\setminus C)\end{cases}. \tag{13}$$

Then π is irreducible as a representation of $\mathbb{F}_\infty$, if and only if $\tilde{\pi}$ is irreducible as a representation of H, and if $\pi_1, \pi_2\in\mathrm{Irr}(\mathbb{F}_\infty)$, then

$$\pi_1 \cong_{\mathbb{F}_\infty} \pi_2 \Leftrightarrow \tilde{\pi}_1 \cong_H \tilde{\pi}_2.$$

Next observation is simple, but will be useful.

Lemma 2.9. *If $l \in \mathbb{Z} \setminus \{0\}$, then $|a^l C a^{-l} \cap C| \leq 1$.*

Proof. Since $c_n = b_{2^n}$, it suffices to shows that there is at most one $n \geq 0$ for which $2^n + l = 2^m$ for some $m \geq 0$. We show this for $l > 0$. Then $m > n$. Assume there is such $n \geq 0$, and let n be smallest such. Assume by contradiction that $n' \geq n+1$ satisfies $2^{n'} + l = 2^{m'}$ for some m', then $m' \geq m+1$, and

$$l = 2^m - 2^n = 2^n(2^{m-n} - 1) = 2^n(2^{m'-n} - 2^{n'-n}),$$

whence $2^{m-n} - 1 = 2^{m'-n} - 2^{n'-n}$, which is a contradiction because the left hand side is odd and the right hand side is even. $l < 0$ case is similar. □

Lemma 2.10. *Let $\pi, \sigma \in \mathrm{Irr}_H(\mathbb{F}_\infty)$. Then $\pi \ncong \sigma^{a^l}$ for every $l \in \mathbb{Z} \setminus \{0\}$.*

Proof. Assume by contradiction that there exists $l \neq 0$ such that $\pi \cong \sigma^{a^l}$. Then there exists $u \in \mathcal{U}(\mathcal{H}_\infty)$ such that $\pi(h) = u\sigma(a^l h a^{-l})u^*$ for all $h \in H$. Let $s \in S = \{a^n b a^{-n}\}$. If $s \in S \setminus C$, then by construction (see (13)) $\pi(s) = 1$. Therefore if $\pi(s) \neq 1$, then $s \in C$. In this case, $\sigma(a^l s a^{-l}) = u^*\pi(s)u \neq 1$ because u is unitary. Therefore by the same reason $a^l s a^{-l} \in C$. By Lemma 2.9, this shows that there is at most one $s \in S$ for which $\pi(s) \neq 1$. Since S generates H, we see that $\pi(H)$ is commutative. Thus by Schur's lemma, π is a 1-dimensional representation, a contradiction ($\dim(\mathcal{H}_\infty) = \infty$). □

Corollary 2.11. *If $\pi \in \mathrm{Irr}(H, \mathcal{H}_\infty)$, then $\mathrm{ind}_H^{\mathbb{F}_2}(\pi)$ is irreducible.*

Proof. By Proposition 2.4 (ii), it suffices to show that $\pi^g \ncong \pi$ for $g \in \mathbb{F}_2 \setminus H$. We observe that if $g' \in Hg$, then $\pi^g \cong \pi^{g'}$: indeed, if $g' = h'g$ for $h' \in H$, then

$$\pi^{g'}(h) = \pi(h'ghg^{-1}(h')^{-1}) = \pi(h')\pi^g(h)\pi(h')^*, \quad h \in H,$$

whence $\pi^g \cong \pi^{g'}$. Therefore since $\{a^l; l \in \mathbb{Z}\}$ is a coset representative of H in $\mathbb{F}_2$, it suffices to show that $\pi^{a^l} \ncong \pi$ for $l \in \mathbb{Z} \setminus \{0\}$. Then this is proved by Lemma 2.10 (put $\sigma = \pi$). □

Theorem 2.12. *The map $\mathrm{Irr}(\mathbb{F}_\infty, \mathcal{H}_\infty) \ni \pi \mapsto \Theta_\pi = V_\pi \mathrm{ind}_H^{\mathbb{F}_2}(\tilde{\pi})V_\pi^{-1} \in \mathrm{Irr}(\mathbb{F}_2, \mathcal{H}_\infty^{\oplus \mathbb{Z}})$ gives a Borel reduction of $\approx_{\mathbb{F}_\infty}$ to $\approx_{\mathbb{F}_2}$. In particular, $\mathbb{F}_2$ is representation universal. Here, we use V_π from Proposition 2.4 (iii).*

Corollary 2.13. *$\mathbb{F}_n$ ($n \in \{2, 3, \cdots, \infty\}$) is representation universal.*

Proof. Since $\mathbb{F}_n$ ($n \geq 2$) surjects onto $\mathbb{F}_2$, the result follows from Theorem 2.12. □

Proof of Theorem 2.12. Let $\pi, \sigma \in \mathrm{Irr}_\infty(\mathbb{F}_\infty)$. Then $\theta_\pi, \sigma_\pi \in \mathrm{Irr}_\infty(\mathbb{F}_2)$, and $\pi \cong_{\mathbb{F}_\infty} \sigma$ implies that $\theta_\pi \cong_{\mathbb{F}_2} \theta_\sigma$ (see Lemma 2.2). Assume that $\theta_\pi \cong \theta_\sigma$. Then restricting to the normal subgroup and by Proposition 2.4, we have

$$\bigoplus_{l \in \mathbb{Z}} \pi^{a^l} \cong \theta_\pi|_H \cong \theta_\sigma|_H \cong \bigoplus_{l \in \mathbb{Z}} \sigma^{a^l}$$

Note that $\tilde{\pi} = \bigoplus_{l \in \mathbb{Z}} \pi^{a^l}$ and $\tilde{\sigma} = \bigoplus_{l \in \mathbb{Z}} \sigma^{a^l}$ are defined on the direct sum Hilbert space $\mathcal{K} = \bigoplus_{l \in \mathbb{Z}} \mathcal{H}_l$, $\mathcal{H}_l = \mathcal{H}_\infty$ ($l \in \mathbb{Z}$). Therefore there exists a unitary $V \colon \mathcal{K} \to \mathcal{K}$ such that $V\tilde{\pi}(h) = \tilde{\sigma}(h)V, \quad h \in H$.

Now we show that there exists $l \in \mathbb{Z}$ such that $\mathrm{Hom}_H(\pi, \sigma^{a^l}) \neq \{0\}$. Once we have this, then since π, σ^{a^l} are irreducible, (a corollary of) Schur's lemma shows that $\pi \cong \sigma^{a^l}$. Then $l = 0$ by Lemma 2.10, and we get $\pi \cong_H \sigma$, which implies $\pi \cong_{\mathbb{F}_\infty} \sigma$, and we are done.

To show the claim, define for each $\xi \in \mathcal{H}_0$ a vector $\tilde{v} = (v_l)_{l \in \mathbb{Z}} \in \mathcal{K}$ by $v_0 = v, v_l = 0$ ($l \neq 0$). Then we define $V_l \colon \mathcal{H}_0 \to \mathcal{H}_l$ by

$$V_l v = (V\tilde{v})_l \in \mathcal{H}_l, \quad v \in \mathcal{H}_0.$$

Then for each $h \in H, v \in \mathcal{H}_0$, we have

$$\begin{aligned} V_l \pi(h) v &= (V\tilde{\pi}(h)\tilde{v})_l = (\tilde{\sigma}(h)V\tilde{v})_l \\ &= \sigma^{a^l}(V\tilde{v})_l = \sigma^{a^l}(h)V_l v. \end{aligned}$$

This shows that $V_l \in \mathrm{Hom}_H(\pi, \sigma^{a^l})$. Since V is a unitary, there exists $l \in \mathbb{Z}$ such that $V_l \neq 0$. Thus $\mathrm{Hom}_H(\pi, \sigma^{a^l}) \neq 0$ for at least one l[2]. $\square$

[2]Since $\sigma^{a^l} \not\cong \sigma^{a^{l'}}$ if $l \neq l'$, there is one and only one l for which this holds. See the proof of Lemma 2.10.