Weyl-von Neumann Theorem and Borel complexity of unitary equivalence modulo compacts of self-adjoint operators

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Joint work with Yasumichi Matsuzawa
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Outline of Talk

Quick reminder:

- An densely defined operator on H is a linear map $A\colon \mathbf{dom}(A) \to H$, where $\mathbf{dom}(A) \subset H$ is a dense subspace.

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- Write $A = B$ if $A \subset B$ and $B \subset A$.

Example

$A = i\frac{d}{dt}, H = L^2[0,1], \mathrm{dom}(A) = AC_0^1[0,1] \subset H$, where

$$AC_0^1[0,1] := \{\varphi \in AC[0,1]; \varphi(0) = \varphi(1) = 0\}.$$

Then $B = i\frac{d}{dt}, \mathrm{dom}(B) = \{\varphi \in AC[0,1]; \varphi(0) = \varphi(1)\}$ is an extension of A.

The adjoint A^* of A is defined as:

$$\mathbf{dom}(A^*) = \{\xi;\ \exists! \zeta \langle \xi, A\eta \rangle = \langle \zeta, \eta \rangle,\ \eta \in \mathbf{dom}(A)\},$$
$$A^*\xi := \zeta.$$

The adjoint A^* of A is defined as:

$$\begin{aligned}\mathrm{dom}(A^*) &= \{\xi;\ \exists!\zeta\langle\xi, A\eta\rangle = \langle\zeta,\eta\rangle,\ \eta\in\mathrm{dom}(A)\},\\ A^*\xi &:= \zeta.\end{aligned}$$

- A is symmetric if $A\subset A^*$ i.e.,

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Example (cont'd)

$A = i\frac{d}{dt}, H = L^2[0,1], \mathrm{dom}(A) = AC_0^1[0,1]\subset H$
A is symmetric, not self-adj: for $\varphi,\psi\in C_0^1[0,1]$,

$$\langle\varphi, A\psi\rangle = \int_0^1 \overline{\psi} i\frac{d\varphi}{dt}dt = i\overline{\psi}\varphi|_0^1 - i\int_0^1\overline{\frac{d\psi}{dt}}\varphi dt$$
$$= \int_0^1 \overline{i\frac{d\psi}{dt}}\varphi dt = \langle A\varphi,\psi\rangle.$$

We fix $H \cong \ell^2$, $H_1 := \{\xi \in H; \|\xi\| \leq 1\}$.

- $\mathbf{SA}(H)$=all s.a. operators.
- $\mathbb{B}(H)_{\mathrm{sa}}$=bounded s.a. operators.
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Remark

If $\boldsymbol{A}$ is s.a. then $\boldsymbol{\sigma}(\boldsymbol{A})$ is always closed and nonempty.

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Theorem (von Neumann,Dixmier)

Let A be unbounded self-adjoint. Then $\exists u \in \mathcal{U}(H)$ s.t.

$$\mathrm{dom}(A) \cap u \cdot \mathrm{dom}(A) = \{0\}.$$

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Good news: $A + B$ is self-adjoint if $A \in \mathbf{SA}(H)$, $B \in \mathbb{B}(H)_{\mathbf{sa}}$.

Essential Spectrum

Definition

For A: self-adj on H, the essential spectra $\sigma_{\mathrm{ess}}(A)$ is the set of all $\lambda \in \sigma(A)$ which is either

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Example

(1) A=multiplication operator on $L^2([0,1])$ I.e.,
$(Af)(x) = xf(x), \quad f \in L^2[0,1]$.
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(3) $A = \sum_{n=1}^{\infty} \frac{1}{n}\langle \xi_n, \cdot \rangle \xi_n$, where $\{\xi_n\}_{n=1}^{\infty}$: CONS.
$\rightsquigarrow \sigma(A) = \{\frac{1}{n}; n \in \mathbb{N}\} \cup \{0\},\ \sigma_{\rm ess}(A) = \{0\}$.

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$\sigma_{\text{ess}}(A)$ *is always closed.*
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Theorem (Weyl)

Let $A \in \mathbf{SA}(H)$, and $K \in \mathbb{K}(H)_{\mathrm{sa}}$. Then $\sigma_{\mathrm{ess}}(A + K) = \sigma_{\mathrm{ess}}(A)$.

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Theorem (Weyl-von Neumann (1))

Let $A \in \mathbf{SA}(H)$ and $\varepsilon > 0$. Then $\exists K \in \mathbb{K}(H)_{\mathrm{sa}}$ with $\|K\| < \varepsilon$ s.t. $A + K$ is diagonal:

$$A + K = \sum_{n=1}^{\infty} a_n \langle \xi_n, \cdot \rangle \xi_n,$$

for some $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and a CONS $\{\xi_n\}_{n=1}^{\infty}$.

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Based on W-vN (1), von Neumann showed [(1)⇒(2) below] that essentially converse to Weyl's Compact Perturbation Theorem is true for bounded s.a. operators:

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Theorem (Weyl-von Neumann(2))

Let $A, B \in \mathbb{B}(H)_{\mathrm{sa}}$. *TFAE*

(1) $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(B)$.

(2) $\exists u \in \mathcal{U}(H)\ \exists K \in \mathbb{K}(H)_{\mathrm{sa}}$ *s.t.*

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Question

Let $A, B \in \mathrm{SA}(H)$ *be s.t.* $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(B)$*. Is it the case that* $uAu^* + K = B$ *for some* u, K*?*

Definition

$A, B \in \mathbf{SA}(H)$ are Weyl-von Neumann equivalent, if $\exists u \in \mathcal{U}(H)$ $\exists K \in \mathbb{K}(H)_{\mathbf{sa}}$ s.t. $uAu^* + K = B$.

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Answer. NO: the question is too optimistic!

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Example

Fix a CONS $\{\xi_n\}_{n=1}^{\infty}$ for H_0, and let $A_0 := \sum_{n=1}^{\infty} n\langle \xi_n, \cdot \rangle \xi_n \in \mathbf{SA}(H_0)$, and define $A, B \in \mathbf{SA}(H)$ by

$$A := A_0 \oplus 0, \quad B := 0 \oplus 0.$$

Then $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(B) = \{0\}$, and since A is unbounded, so is $uAu^* + K$ for any $u \in \mathcal{U}(H)$ and $K \in \mathbf{SA}(H)$.
Thus $uAu^* + K \neq B$.

So the answer is NO. But still we may find a stronger invariant to classify A, B up to W-vN equivalence?

Observation

If $A \overset{\text{W-vN}}{\sim} B$, then their domains must be unitarily equivalent:
$\exists u \in \mathcal{U}(H)$ *s.t.* $u \cdot \mathrm{dom}(A) = \mathrm{dom}(B)$.

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Example

Let $\{\xi_n\}_{n=0}^{\infty}$ be a CONS for H, and define $\{A_t\}_{t\in(0,1)} \subset \mathrm{SA}(H)$ by

$$A_t = \sum_{n=1}^{\infty} 2^{n^t} \langle \xi_n, \cdot \rangle \xi_n \quad (0 < t < 1).$$

We can show (using operator ranges) that $\{A_t\}_{t\in(0,1)}$ satisfies $\sigma_{\mathrm{ess}}(A_t) = \emptyset$ $(0 < t < 1)$, but $\mathrm{dom}(A_t)$ and $\mathrm{dom}(A_s)$ are not unitarily equivalent for $0 < t \neq s < 1$.

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Let $A, B \in \mathbf{SA}(H)$ *be s.t.* $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(B)$ *and* $\mathbf{dom}(A), \mathbf{dom}(B)$ *are unitarily equivalent. Are* A *and* B *W-vN equivalent?*

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Answer

Again NO.

We can find another continuous family $\{B_t\}_{t\in[0,1]}$ with

(1) $\mathbf{dom}(B_t)$ are the same $\forall t \in [0,1]$.

(2) $\sigma_{\rm ess}(B_t) = \mathbb{N}\ \forall t \in [0,1]$.

(3) No two of them are W-vN equivalent.

Example

Fix a bijection $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$ given by

$$\langle k, m \rangle := 2^{k-1}(2m-1), \qquad m, k \in \mathbb{N}.$$

and define $\{B_t\}_{t \in [0,1]}$ by

$$B_t := \sum_{n=1}^{\infty} \lambda_n^{(t)} \langle \xi_n, \ \cdot \ \rangle \xi_n, \quad \lambda_{\langle k,m \rangle}^{(t)} := k + \frac{t}{m+2}.$$

$\rightsquigarrow$ $\{B_t\}_{t \in [0,1]}$ have the same domain and ess. spec$=\mathbb{N}$.
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$\rightsquigarrow$ **Descriptive Set Theory** tells us that it is indeed impossible.

Strategy: introduce a Polish topology (called SRT) on $\mathbf{SA}(H)$, and regard W-vN equivalence as the orbit equivalence $E_G^{\mathbf{SA}(H)}$, where $G = \mathbb{K}(H)_{\mathrm{sa}} \rtimes \mathcal{U}(H) \curvearrowright \mathbf{SA}(H)$ by

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Outline:

- $\mathbf{SA}(H)$ is a Polish G-space. $\mathbb{B}(H)_{\mathrm{sa}}$ is a Borel G-inv. subset.
- Show that $E_G^{\mathbf{SA}(H)}$ is considerably more complex than $E_G^{\mathbb{B}(H)_{\mathrm{sa}}}$ (w.r.t. $\leq_B$).

Definition

Let $\boldsymbol{E}, \boldsymbol{F}$ be equiv. relations on Polish (st Borel) spaces $\boldsymbol{X}, \boldsymbol{Y}$, resp. We say

(1) $\boldsymbol{E}$ is Borel reducible to $\boldsymbol{F}$, $\boldsymbol{E} \leq_{\boldsymbol{B}} \boldsymbol{F}$, if there is a Borel map $\boldsymbol{f} \colon \boldsymbol{X} \to \boldsymbol{Y}$ s.t.

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Equivalently, if $E \leq_B E_{S_\infty}^{X}$ for some Polish S_∞-space X.

Definition (Hjorth)

Let G be a Polish group and X a Polish G-space.

(1) Let $x \in X$. For $(x \in)U \overset{\text{open}}{\subset} X$ and $(1 \in)V \overset{\text{open}}{\subset} G$, the *local U-V orbit* of x, denoted $\mathcal{O}(x, U, V)$, is the set of all $y \in U$ for which $\exists l \in \mathbb{N}$, $x = x_0, x_1, \cdots, x_l = y \in U$, and $\exists g_0, \cdots, g_{l-1} \in V$, s.t. $x_{i+1} = g_i \cdot x_i$ for all $0 \leq i < l$.

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Theorem (Hjorth)

Let X be a Polish G-space with every orbit meager and some orbit dense. TFAE:

(1) *X is generically turbulent.*

(2) *For any Borel S_∞-space Y, E_G^X is generically $E_{S_\infty}^Y$-ergodic.*

Strategy: Consider $E_G^{\mathbf{SA}(H)}$, where $G = \mathbb{K}(H)_{\mathbf{sa}} \rtimes \mathcal{U}(H) \curvearrowright \mathbf{SA}(H)$ by

$$(K, u) \cdot A := uAu^* + K, \qquad K \in \mathbb{K}(H)_{\mathbf{sa}},\ u \in \mathcal{U}(H).$$

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Main Theorem and Strategy

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(♣1) $G \curvearrowright \mathbf{SA}(H)$ has a comeager orbit, so we cannot use Hjorth Turbulence Theorem directly.

(♣2) But we can find another turbulent $G \curvearrowright Y$ s.t. $E_G^Y \leq_B E_G^{\mathbf{SA}(H)}$.

Polish Space $\mathrm{SA}(H)$

Definition

The Strong Resolvent Topology (SRT) on $\mathbf{SA}(H)$ is the weakest topology for which $\mathbf{SA}(H) \ni A \mapsto (A-i)^{-1} \in \mathbb{B}(H)$ is continuous w.r.t. strong operator topology (SOT).

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Let $\{\xi_n\}_{n=1}^{\infty}$: CONS for H. One might want to define d' by

$$d'(A,B) := \sum_{n=1}^{\infty} \frac{1}{2^n} \|(A-i)^{-1}\xi_n - (B-i)^{-1}\xi_n\|.$$

d' is a metric compatible with SRT. However, d' is not complete.

Instead, we use:

Theorem (Trotter)

Let $A_n, A \in \mathbf{SA}(H)$. *Then* $A_n \to A$ *(SRT)*$\Leftrightarrow$ $e^{itA_n} \to e^{itA}$ *(SOT) compact uniformly in* t.

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Proof.

Separability is easy. By Trotter's Theorem, the metric d given by

$$d(A,B) := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{2^{n+m}}\sup_{t\in[-m,m]}\|e^{itA}\xi_n - e^{itB}\xi_n\|,$$

is compatible with SRT, and it can be shown to be complete. □

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The proof is done by Neumann series method. Note that $\mathbf{SA}(H)$ is not a vector space : "addition $(A, B) \mapsto A + B$" is not well-defined in $\mathbf{SA}(H)$.

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Let $F_n := \{A \in \mathbb{B}(H)_{\mathrm{sa}}; \|A\| \leq n\}$. Then $\mathbb{B}(H)_{\mathrm{sa}} = \bigcup_{n=1}^{\infty} F_n$, and F_n is SRT-closed.

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Then $A_k := \sum_{m=1}^{k} a_m e_m + \sum_{m=k+1}^{\infty} m e_m \overset{k\to\infty}{\to} A_0$ (SRT).

$\rightsquigarrow A_k \in F_n$ eventually, a contradiction (A_k is unbounded)! □

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Corollary (AM '14)

$E_G^{\mathbb{B}(H)_{\mathrm{sa}}}$ is smooth.

Sketch: $A \mapsto \sigma_{\mathrm{ess}}(A)$ is Borel.

Step1: $\mathbf{SA}(H) \ni A \mapsto \sigma(A) \in \mathcal{F}(\mathbb{R})$ is Borel. Easy by Stone formula:

$$E_A([a,b]) + E_A((a,b)) = \lim_{\varepsilon \to +0} \frac{1}{\pi i} \int_a^b \{(A-(t+i\varepsilon))^{-1} - (A-(t-i\varepsilon))^{-1}\} dt,$$

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$$\{A; \sigma_{\mathrm{ess}}(A) \cap [a,b] \neq \emptyset\} = \left\{A; \bigcap_{n=1}^{\infty} \sigma(A+K_n) \cap [a,b] \neq \emptyset\right\}.$$

$$\overset{(*)}{=} \bigcap_{N=1}^{\infty} \underbrace{\left\{A; \bigcap_{n=1}^{N} \sigma(A+K) \cap [a,b] \neq \emptyset\right\}}_{=:\mathcal{B}_N}.$$

($(*)$: compactness of $[a,b]$) $\rightsquigarrow$ enough to show that $\mathcal{B}_N$ is Borel ($\forall N$).

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Therefore $\Psi_n\colon \mathrm{SA}(H) \to \mathcal{F}(\mathbb{R})$ given by

$$\Psi_n = I_n \circ (\sigma \circ \tau_1 \times \cdots \times \sigma \circ \tau_n) : A \mapsto \bigcap_{k=1}^n \sigma(A+K_k)$$

is Borel, whence $\mathcal{B}_n = \Psi_n^{-1}(\{F \in \mathcal{F}(\mathbb{R}); F \cap [a,b] \neq \emptyset\})$ is Borel. □

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is Borel, whence $\mathcal{B}_n = \Psi_n^{-1}(\{F \in \mathcal{F}(\mathbb{R}); F \cap [a,b] \neq \emptyset\})$ is Borel. □

Therefore $E_G^{\mathbb{B}(H)_{\mathrm{sa}}}$ is smooth.

Next we show that $E_G^{\mathrm{SA}(H)}$ is unclassifiable by countable structures.

Definition (Hjorth)

Let G be a Polish group and X a Polish G-space.

(1) Let $x \in X$. For $(x \in) U \overset{\text{open}}{\subset} X$ and $(1 \in) V \overset{\text{open}}{\subset} G$, the *local U-V orbit* of x, denoted $\mathcal{O}(x, U, V)$, is the set of all $y \in U$ for which $\exists l \in \mathbb{N}$, $x = x_0, x_1, \cdots, x_l = y \in U$, and $\exists g_0, \cdots, g_{l-1} \in V$, s.t. $x_{i+1} = g_i \cdot x_i$ for all $0 \leq i < l$.

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(2) The action is *turbulent* at $x \in X$ if the local orbits $\mathcal{O}(x, U, V)$ are somewhere dense $(x \in)\forall U \overset{\text{open}}{\subset} X$ and $(1 \in)\forall V \overset{\text{open}}{\subset} G$.

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Theorem (Hjorth)

Let X be a Polish G-space with every orbit meager and some orbit dense. TFAE:

(1) *X is generically turbulent.*

(2) *For any Borel S_∞-space Y, E_G^X is generically $E_{S_\infty}^Y$-ergodic.*

STEP1: $G \curvearrowright \mathbf{SA}(H)$ has a comeager orbit:

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Remark

It can be shown in cotrast that the action of the subgroup $\mathbb{K}(H)_{\mathrm{sa}} \curvearrowright \mathbf{SA}(H)$ *is genrically turbulent.*

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$$\begin{aligned} G_\lambda^c &= \{A \in \mathrm{SA}(H);\ \lambda \notin \sigma_{\mathrm{ess}}(A)\} \\ &= \bigcup_{\varepsilon>0} \{A \in \mathrm{SA}(H);\ 1_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A) \text{ is of finite rank }\} \\ &= \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \underbrace{\{A \in \mathrm{SA}(H); \mathrm{rank}(1_{(\lambda-\frac{1}{n},\lambda+\frac{1}{n})}(A)) \leq k\}}_{=:S_{n,k}} \end{aligned}$$

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We can show that $S_{n,k}$ is closed and nowhere-dense in $\mathrm{SA}(H)$. Thus G_λ is dense G_δ, and so is

$$\mathrm{SA}(H)_{\mathrm{full}} = \{A; \sigma_{\mathrm{ess}}(A) = \mathbb{R}\} = \bigcap_{q \in \mathbb{Q}} G_q$$

by Baire category Theorem ($\sigma_{\mathrm{ess}}(A)$ is closed!). □

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$$A = \sum_{n=1}^{\infty} a_n \langle \xi_n, \cdot \rangle \xi_n, B = \sum_{n=1}^{\infty} b_n \langle \eta_n, \cdot \rangle \eta_n,$$

where $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ dense in $\mathbb{R}$.

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Define $u \in \mathcal{U}(H)$ by

$$u\xi_k := \eta_{\pi^{-1}(k)}, \quad k \in \mathbb{N}.$$

Then $uAu^* = \sum_{n=1}^{\infty} a_{\pi(n)} \langle \eta_n, \cdot \rangle \eta_n$. Therefore $uAu^* + K = B$, where $K := \sum_{n=1}^{\infty} (b_n - a_{\pi(n)}) \langle \eta_n, \cdot \rangle \eta_n \in \mathbb{K}(H)_{\text{sa}}$. .

□

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The *norm resolvent topology* (NRT) on $\mathbf{SA}(H)$ is the weakest topology for which $\mathbf{SA}(H) \ni A \mapsto (A - i)^{-1} \in \mathbb{B}(H)$ is norm-continuous.

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Proposition (AM '14)

$(\mathbf{EES}(H), \mathbf{NRT})$ *is a Polish G-space with respect to the restriction of the G-action on $\mathbf{SA}(H)$.*

Theorem (AM '14)

$G \curvearrowright \mathbf{EES}(H)$ *is generically turbulent, and* $E_G^{\mathbf{EES}(H)} \le_B E_G^{\mathbf{SA}(H)}$.

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Sketch (meagerness of orbits).

We show $[A]_G$ is meager for all $A \in \mathbf{EES}(H)$.

Step 1. If $\lambda \in \sigma(A), K \in \mathbb{K}(H)_{\mathrm{sa}}, c > 1$, then

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Step 3. $\mathcal{S}_q := \bigcap_{\lambda \in \sigma_{\mathrm{p}}(B)} S_{q,\lambda}$ is closed nowhere-dense (non-trivial). □

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$A := A_0 \oplus 0, B := 0 \oplus 0 \in \mathrm{SA}(H^{\oplus 2}), A_0 = \sum_{n=1}^{\infty} ne_n$ satisfy
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$$(A-i)^{-1} - u(B-i)^{-1}u^* = [(A_0 - i1_{H_0})^{-1} - i1_H] \oplus 0 \notin \mathbb{K}(H^{\oplus 2}),$$

because $(A_0 - i1_H)^{-1} \in \mathbb{K}(H^{\oplus 2})_{\mathrm{sa}}$ and $1_H \notin \mathbb{K}(H)$.

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$\rightsquigarrow A, B$ are not $\overset{\mathbf{WvN2}}{\sim}$-equivalent.

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So

$$\overline{\sigma}_{\text{ess}} : \mathrm{SA}(H) \to \mathcal{F}(\mathbb{R}) \times \{0,1\} =: X$$

$$\overline{\sigma}_{\text{ess}}(A) := \begin{cases} (\sigma_{\text{ess}}(A), 0) & (A \text{ is bounded }) \\ (\sigma_{\text{ess}}(A), 1) & (A \text{ is unbounded }) \end{cases}.$$

gives us a Borel reduction $\overset{\mathbf{WvN2}}{\sim} \leq_B \mathrm{id}_X$. □

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Question

Is $\sim_{\mathbf{dom}}$ *Borel?*

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$$\begin{aligned}\xi\in\mathrm{dom}(A) &\Leftrightarrow \exists\lim_{t\to 0}\frac{1}{t}(e^{itA}\xi-\xi)\\ &\Leftrightarrow \forall\varepsilon\in\mathbb{Q}_+\ \exists\delta\in\mathbb{Q}_+\ \text{s.t. } \forall s,t\in\mathbb{Q}^{\times}\cap(-\delta,\delta),\\ &\qquad \left\|\frac{e^{isA}\xi-\xi}{s}-\frac{e^{itA}\xi-\xi}{t}\right\|\le\varepsilon.\end{aligned}$$

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