Ultraproducts, QWEP von Neumann Algebras, and Effros-Maréchal Topology

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Joint work with Uffe Haagerup and Carl Winsløw
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Outline of Talk

1. Kirchberg's QWEP Conjecture
2. Effros-Maréchal Topology
3. Ultraproduct of von Neumann algebras
4. Characterizations of QWEP von Neumann Algebras

- H. Ando, U. Haagerup, "Ultraproucts of von Neumann algebras", arXiv:1212.5457
- H. Ando, U. Haagerup, C. Winsløw, "Ultraproducts, QWEP von Neumann algebras, and the Effros-Maréchal topology", arXiv:1306.0460

QWEP Conjecture

Kirchberg ('93) revealed remarkable connetions among

- Tensor products of C*-algebras
- Lance's Weak Expectation Property (WEP)
- Connes's Embedding Conjecture (CEC): $\forall \boldsymbol{N}$ sep. II_1 factor embeds into $\boldsymbol{R}^{\omega}$?

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In this talk, we discuss how QWEP property is connected to ultraproducts of von Neumann algebras using topological method.

QWEP Conjecture

Definition (Lance '73, Kirchberg '93)

(1) C*-alg A has the weak expectation property (WEP) if for any faithful representation $A \subset \mathbb{B}(H)$, there is a ucp map $\Phi\colon \mathbb{B}(H) \to A^{**}$ s.t. $\Phi|_A = \mathrm{id}_A$.

(2) C*-alg A has the quotient weak expectation property (QWEP) if it is the quotient of a C*-algebra with WEP.

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Theorem (Kirchberg's QWEP Conjecture)

TFAE.

(1) $C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$.

(2) *Every C^*-algebra has QWEP.*

(3) $C^*(\mathbb{F}_\infty)$ *has WEP.*

(4) *(Connes's Embedding Conjecture) Every separable type II_1 factor M admits an embedding into R^ω, where R is the hyperfinite II_1 factor.*

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Is QWEP conjecture true?

$$C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) \stackrel{?}{=} C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$$

Kirchberg('93) proved

$$C^*(\mathbb{F}_\infty) \otimes_{\min} \mathbb{B}(\ell^2) = C^*(\mathbb{F}_\infty) \otimes_{\max} \mathbb{B}(\ell^2).$$

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Theorem (Junge-Pisier '95)

$\mathbb{B}(\ell^2) \otimes_{\min} \mathbb{B}(\ell^2) \neq \mathbb{B}(\ell^2) \otimes_{\max} \mathbb{B}(\ell^2)$.

Effros-Maréchal Topology

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- Maréchal ('73) introduced Polish topology on vN(H) that generates Effros Borel structure.
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Definition (Maréchal '73)

The Effros-Maréchal Topology on vN($\boldsymbol{H}$) is the weakest topology which makes all the maps of the form

$$\mathrm{vN}(\boldsymbol{H}) \ni \boldsymbol{M} \mapsto \|\varphi|_{\boldsymbol{M}}\|, \quad \varphi \in \mathbb{B}(\boldsymbol{H})_*$$

continuous.

Definition (Haagerup-Winsløw '98)

For $\{M_n\}_{n=1}^{\infty} \subset \mathrm{vN}(H)$, define $\limsup_{n\to\infty} M_n$ and $\liminf_{n\to\infty} M_n$ by

(1) $\liminf_{n\to\infty} M_n = \{x \in \mathbb{B}(H);\ x_n \overset{\mathrm{so}*}{\to} x, \exists (x_n)_n \in \ell^{\infty}(\mathbb{N}, M_n)\}$.

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Theorem (Haagerup-Winsløw '98)

TFAE.

(1) $M_n \to M$ *in* vN(H).

(2) $\liminf\limits_{n\to\infty} M_n = M = \limsup\limits_{n\to\infty} M_n$.

Moreover, $\left(\limsup\limits_{n\to\infty} M_n\right)' = \liminf\limits_{n\to\infty} M_n'$ *holds.*

Important subsets of vN$(\boldsymbol{H})$: $\mathcal{F}$: factors, $\mathcal{F}_{\mathbf{inj}}$ injective factors, vN$(\boldsymbol{H})^{\mathbf{st}}$ standardly acting vNas, $\mathcal{F}_{\mathbf{II_1}}$ type $\mathsf{II}_{\mathbf{1}}$ factors, etc.

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Theorem (Haagerup-Winsløw '00)

Subset of vN(H)	*Dense in vN(H)?*	*G_δ?*
$\mathcal{F}$	*Yes*	*Yes*
$\bigcup_{n\le n_0} \mathcal{F}_{\mathrm{I}_n},\ n_0 \in \mathbb{N}$	*No*	*Yes (closed)*
$\mathcal{F}_{\mathrm{I}_{\mathrm{fin}}}$	*	*No (but F_σ)*
$\mathcal{F}_{\mathrm{I}_\infty}$	*	*No (but F_σ)*
$\mathcal{F}_{\mathrm{II}_1}$	*Yes*	*No*
$\mathcal{F}_{\mathrm{II}_\infty}$	*Yes*	*No*
$\mathcal{F}_{\mathrm{III}_0}$	*Yes*	*No*
$\mathcal{F}_{\mathrm{III}_\lambda},\ \lambda \in (0,1)$	*Yes*	*No*
$\mathcal{F}_{\mathrm{III}_1}$	*Yes*	*Yes*
$\mathcal{F}_{\mathrm{inj}}$	*	*Yes*
$\mathcal{F}^{\mathrm{st}}$	*Yes*	*Yes*

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$\mathcal{F}_{\mathrm{III}_\lambda},\ \lambda\in(0,1)$	*Yes*	*No*
$\mathcal{F}_{\mathrm{III}_1}$	*Yes*	*Yes*
$\mathcal{F}_{\mathrm{inj}}$	*	*Yes*
$\mathcal{F}^{\mathrm{st}}$	*Yes*	*Yes*

*Moreover, * are all equivalent to QWEP (Connes Embedding) conjecture.*

Quick reminder:

Definition

$(M, \mathcal{H}, J, \mathcal{P}_M^\natural)$ is called a standard form of M if $J : \mathcal{H} \to \mathcal{H}$ is an antilinear involution, $\mathcal{P}_M^\natural = (\mathcal{P}_M^\natural)^0$ self-dual convex cone in $\mathcal{H}$ such that

(1) $JMJ = M'$.

(2) $J\xi = \xi, \quad \xi \in \mathcal{P}_M^\natural$.

(3) $xJxJ\mathcal{P} \subset \mathcal{P}_M^\natural, \quad x \in M$.

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GNS rep $(\pi_\varphi, H_\varphi, \xi_\varphi)$ of f.n. state φ gives a standard form.

In this case, $\mathcal{P}_M^\natural = \overline{\Delta_\varphi^{\frac{1}{4}} M_+ \xi_\varphi}$, $J = J_\varphi$.

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In this case, $\mathcal{P}_M^\natural = \overline{\Delta_\varphi^{\frac{1}{4}} M_+ \xi_\varphi}$, $J = J_\varphi$.$\forall M$, $\exists!$ $(M, J, \mathcal{P}, \mathcal{H})$ (Haagerup '75).

Very useful tricks:

Theorem (Expectation-Trick, Haagerup-Winsløw '00)

Assume $N \subset M$ *be* ∞*-dim vNas on* H*,* N *standard on* H*, and there is faithful normal expectation* $E\colon M \to N$*. Then* $\exists u_n \in \mathcal{U}(H)$ *and* $\exists M_0 \cong M$ *s.t.* $u_n M_0 u_n^* \to N$ *in* $\mathrm{vN}(H)$*.*

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Theorem (⊗-trick, Haagerup-Winsløw'00)

Let $K \cong \ell^2$, *and* $v_0 \in \mathcal{U}(H \otimes K, H)$. *Then* $\exists u_n \in \mathcal{U}(H \otimes K)$ *s.t. for any* $N \in \mathrm{vN}(H)$ *and* $M \in \mathrm{vN}(K)$, *one has*

$$v_0 u_n^* (N \overline{\otimes} M) u_0 v_0^* \overset{n\to\infty}{\to} N \quad \text{in } \mathrm{vN}(H).$$

Very useful tricks:

Theorem (Expectation-Trick, Haagerup-Winsløw '00)

Assume $N \subset M$ be ∞-dim vNas on H, N standard on H, and there is faithful normal expectation $E\colon M \to N$. Then $\exists u_n \in \mathcal{U}(H)$ and $\exists M_0 \cong M$ s.t. $u_n M_0 u_n^* \to N$ in $\mathrm{vN}(H)$.

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$\rightsquigarrow$ Type III factors are dense, McDuff factors are dense in factors $\mathcal{F}$, etc...

Theorem (Haagerup-Winsløw '00)

Let M be a II_1 factor on H. TFAE.

(1) $M \in \overline{\mathcal{F}_{\mathbf{inj}}}$.

(2) *There is an embedding* $i : M \to R^{\omega}$.

Effros-Maréchal Topology

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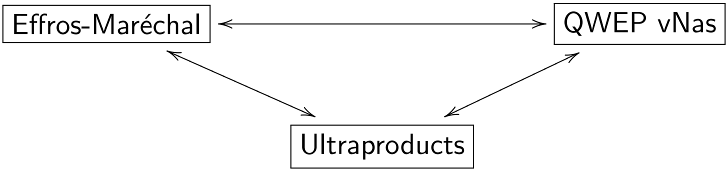
Goal of today's talk: further investigation of $\overline{\mathcal{F}_{\mathbf{inj}}}$ and study the connection of the following:



The Ocneanu Ultaproduct

$(M_n, \varphi_n)_{n=1}^{\infty}$ sequence of vNas/n.f.states. $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$.

Definition (Ocneanu '85)

Define Ocneanu ultraproduct
$(M_n, \varphi_n)^\omega := \mathcal{M}^\omega(M_n, \varphi_n)/\mathcal{I}_\omega(M_n, \varphi_n)$, where

$$\mathcal{I}_\omega(M_n, \varphi_n) := \{(x_n)_n \in \ell^\infty(\mathbb{N}, M_n); \|x_n\|_{\varphi_n}^\sharp \overset{n\to\omega}{\to} 0\},$$
$$\mathcal{M}^\omega(M_n, \varphi_n) := \{x \in \ell^\infty(\mathbb{N}, M_n); x\mathcal{I}_\omega + \mathcal{I}_\omega x \subset \mathcal{I}_\omega\}.$$

$\varphi^\omega : (x_n)^\omega \mapsto \lim_\omega \varphi_n(x_n)$ is a n.f. state on $(M_n, \varphi_n)^\omega$.

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$\varphi^{\omega}:(x_n)^{\omega}\mapsto\lim_{\omega}\varphi_n(x_n)$ is a n.f. state on $(M_n,\varphi_n)^{\omega}$.

Remark

Ocneanu considered the case $M_n\equiv M,\varphi_n\equiv\varphi$. In this case $(M,\varphi)^{\omega}$ is independent of φ, so denote it as M^{ω}.
However, it is crucial to use the sequence $\{\varphi_n\}_{n=1}^{\infty}$ to study Effros-Maréchal topology.

The Groh-Raynaud Ultraproduct

Taking Ocneanu UP does not commute with taking NC L^p-spaces: for $p = 1$, $\mathbb{B}(\ell^2)^\omega = \mathbb{B}(\ell^2)$, but $(\mathbb{B}(\ell^2)_*)_\omega \neq \mathbb{B}(\ell^2)_*$

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Is there a right UP $M \mapsto \widetilde{M}^\omega$ s.t. $(L^p(M))_\omega = L^p(\widetilde{M^\omega})$?

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Theorem (Raynaud '02)

For each $\{M_n\}_{n=1}^\infty$, $!\exists \prod^\omega M_n$ *s.t.*

- $L^p(\prod^\omega M_n) = (L^p(M_n))_\omega$ $(1 \leq p < \infty)$.
  In particular, $(\prod^\omega M_n)_* = ((M_n)_*)_\omega$.
- $(\prod^\omega M_n)' = \prod^\omega M_n'$ *(in standard form)*

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Remark

M^ω *and* $\prod^\omega M$ *are VERY different: in fact (Raynaud)* $\prod^\omega \mathbb{B}(\ell^2)$ *is not semifinite.*

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Remark

M^ω and $\prod^\omega M$ are VERY different: in fact (Raynaud) $\prod^\omega \mathbb{B}(\ell^2)$ is not semifinite. Moreover (AH'12), while R^ω is a II_1 factor, $\prod^\omega R$ is neither semifnite nor a factor!

Ocneanu vs Groh-Raynaud

$M^\omega \neq \prod^\omega M$ in many ways. But are they unrelated?
Consider $\{M_n, \varphi_n\}_{n=1}^\infty$. Then
$\varphi_\omega = (\varphi_n)_\omega \in ((M_n)_*)_\omega = (\prod^\omega M_n)_*$.
Let $p = \mathrm{supp}(\varphi_\omega) \in \prod^\omega M_n$.

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1. $\sigma_t^{(\varphi_n)^\omega}((x_n)^\omega) = (\sigma_t^{\varphi_n}(x_n))^\omega$ *for* $(x_n)^\omega \in (M_n, \varphi_n)^\omega$.
2. *If M is a type III$_\lambda$ $(\lambda \neq 0)$ factor, so are $M^\omega, \prod^\omega M$.*
3. *If M is a type III$_0$ factor, then M^ω is never a factor.*

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$p(\prod^\omega M_n)p \cong (M_n, \varphi_n)^\omega$ *holds. Moreover,*

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(6) $M_\omega \neq M' \cap M^\omega$ *in general, but* $M_\omega = \mathbb{C} \Rightarrow M' \cap M^\omega = \mathbb{C}$ *if* M_* *separable (solution to Ueda's problem).*

Connes's Embedding for general von Neumann algebras?

Approximation Theorem

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For vNas M, N, we write $N \overset{\leftarrow \varepsilon}{\hookrightarrow_i} M$ if there is an embedding $i : N \to M$ and a normal faithful conditional expectation $\varepsilon : M \to i(N)$.

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Remark

- *If $M \overset{\leftarrow \varepsilon}{\hookrightarrow}_i \mathbb{B}(\ell^2)$, then M is atomic.*
- *If $N \overset{\leftarrow \varepsilon}{\hookrightarrow}_i M$ and M is semifinite, then N is also semifinite (Sakai, Tomiyama).*

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Suppose $M_n \to N$ *in* $\mathrm{vN}(H)$*. Then for any n.f. state* $\chi \in \mathbb{B}(H)_*$*,*

$$N \overset{\leftarrow\ \varepsilon}{\hookrightarrow_i} (M_n, \psi_n)^\omega,$$

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Sketch.

(Construction of i) Let $x \in N = \lim M_n$. Then $N = \liminf M_n$. So $\exists (x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$ s.t. $x_n \overset{\text{so}*}{\to} x$. Then $(x_n)_n \in \mathcal{M}^\omega(M_n, \psi_n)$ and $i\colon N \ni x \mapsto (x_n)^\omega \in (M_n, \psi_n)^\omega$ is well-defined.

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Suppose $M_n \to N$ in $\mathrm{vN}(H)$. Then for any n.f. state $\chi \in \mathbb{B}(H)_*$,

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(Construction of ε) Given $x = (x_n)^\omega \in (M_n, \psi_n)^\omega$, let $\tilde{x} :=$ wo-$\lim_\omega x_n$. Then as $N = \limsup M_n$, $\tilde{x} \in N$ and $\varepsilon \colon (M_n, \psi_n)^\omega \ni x \mapsto i(\tilde{x}) \in i(N)$ is well-defined.
By direct calculations, we have $N \overset{\overleftarrow{\varepsilon}}{\hookrightarrow}_i (M_n, \psi_n)^\omega$. □

Corollary (A-Haagerup-Winsløw '13)

Let $M, N \in \mathrm{vN}(H)$ *and assume* $M_n \cong M$ $(\forall n)$ *and* $M_n \overset{n\to\infty}{\to} N$ *in* $\mathrm{vN}(H)$. *Then* $\exists \{\psi_n\}_{n=1}^{\infty}$ *s.t.*

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As a (partial) converse:

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As a (partial) converse:

Proposition (A-Haagerup-Winsløw '13)

Suppose $M_n,\ N \in \mathrm{vN}(H)^{\mathrm{st}}$ *and* $\exists\psi_n \in S_{\mathrm{nf}}(M_n)$ *s.t.* $N \overset{\leftarrow\,\varepsilon}{\hookrightarrow}_i (M_n, \psi_n)^{\omega}$. *Then* $\exists u_n \in \mathcal{U}(H)$ *and* $\exists n_1 < n_2 < \cdots$ *s.t.*

$$u_{n_k} M_{n_k} u_{n_k}^* \overset{k\to\infty}{\to} N \quad \text{in} \ \mathrm{vN}(H).$$

Sketch.

Put $\widetilde{M} = (M_n, \psi_n)^\omega, \tilde{\psi} = (\psi_n)^\omega, \tilde{\varphi} = \tilde{\psi} \circ \varepsilon$. Then by AH'12, $\exists \varphi_n \in S_{\mathrm{nf}}(M_n)$ s.t. $\widetilde{M} = (M_n, \varphi_n)^\omega$ and $\tilde{\varphi} = (\varphi_n)^\omega$.

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By Haagerup-Winsløw'00, $\exists v_n \in \mathcal{U}(L, H)$ s.t. $\xi = (v_n \xi)_\omega, \ \forall \xi \in L$.

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We can then prove that $v_n^* M_n v_n \stackrel{n \to \omega}{\to} \pi_L(N)$ (hard part). Passing to a subsequence and by the uniqueness of standard form, we are done. □

With additional efforts, we get the main result:

Theorem (A-Haagerup-Winsløw '13)

For $M \in \mathrm{vN}(H)$, *and* $0 < \lambda < 1$. *TFAE.*

(1) $M \in \overline{\mathcal{F}_{\mathbf{inj}}}$.

(2) M *has QWEP.*

(3) $M \overset{\leftarrow \varepsilon}{\hookrightarrow_i} R_\infty^\omega$. R_∞: *hyperfinite* III_1 *factor.*

(4) $M \overset{\leftarrow \varepsilon}{\hookrightarrow_i} R_\lambda^\omega$. R_λ: *hyperfinite* III_λ *factor.*

(5) $M \overset{\leftarrow \varepsilon}{\hookrightarrow_i} (M_{k_n}(\mathbb{C}), \varphi_n)^\omega$ *for some* $\{k_n\}_{n=1}^\infty$ *and* $\varphi_n \in S_{\mathrm{nf}}(M_{k_n}(\mathbb{C}))$.

(6) $\forall \varepsilon > 0$, $\forall n \in \mathbb{N}$, $\forall \xi_1, \cdots, \xi_n \in \mathcal{P}_M^\natural$, $\exists k \in \mathbb{N}$ *and* $\exists a_1, \cdots, a_n \in M_k(\mathbb{C})_+$ *s.t.*

$$|\langle \xi_i, \xi_j \rangle - \mathrm{tr}_k(a_i a_j)| < \varepsilon \qquad (1 \le i, j \le n).$$

Here, $\mathcal{P}_M^\natural$ *is the natural cone in the standard form of* M.

Very useful trick:

Theorem ($\otimes$-trick, Haagerup-Winsløw '00)

Let $K \cong \ell^2$, *and* $v_0 \in \mathcal{U}(H \otimes K, H)$. *Then* $\exists u_n \in \mathcal{U}(H \otimes K)$ *s.t. for any* $N \in \mathrm{vN}(H)$ *and* $M \in \mathrm{vN}(K)$, *one has*

$$v_0 u_n^* (N \overline{\otimes} M) u_0 v_0^* \overset{n \to \infty}{\to} N \quad \text{in } \mathrm{vN}(H).$$

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⇝ Type III factors are dense, McDuff factors are dense in factors $\mathcal{F}$, etc...

Sketch: (1) $M \in \overline{\mathcal{F}_{\text{inj}}}$ $\Leftrightarrow$(3)$M \overset{\leftarrow\ \varepsilon}{\hookrightarrow_i} R_\infty^\omega$ $\Rightarrow$(2) M: QWEP$\Rightarrow$(1).

(1)$\Rightarrow$(3) By HW '00, $\mathcal{F}_{\mathrm{III}_1} \cap \mathcal{F}_{\text{inj}}$ is dense in $\mathcal{F}_{\text{inj}}$, so $\exists M_n \cong R_\infty$ s.t. $M_n \to M$. Then $\exists \psi_n \in S_{\text{nf}}(R_\infty)$ s.t. $N \overset{\leftarrow\ \varepsilon}{\hookrightarrow_i} (R_\infty, \psi_n)^\omega$, but $(R_\infty, \psi_n)^\omega \cong R_\infty^\omega$ by AH'12.

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(3)$\Rightarrow$(1) Assume $M \overset{\leftarrow\,\varepsilon}{\hookrightarrow}_i R_\infty^\omega$. Find K_1, K_2 sep s.t.
$\widetilde{M} = M\overline{\otimes}\mathbb{B}(K_1)\overline{\otimes}\mathbb{C}1_{K_2}$ standard on $H \otimes K$, $K := K_1 \otimes K_2$.

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$\rightsquigarrow \widetilde{M} \overset{\leftarrow\,\varepsilon'}{\hookrightarrow}_{i'} Q^\omega$, $Q := R_\infty \overline{\otimes} \mathbb{B}(K_1) \overline{\otimes} \mathbb{C}1_{K_2}$. Since $Q, \widetilde{M}$: standard, $\exists w_n$ s.t. $w_n^* Q w_n \to \widetilde{M}$, so $\widetilde{M} \in \overline{\mathcal{F}_{\mathrm{inj}}}$.

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$\exists w_n$ s.t. $w_n^* Q w_n \to \widetilde{M}$, so $\widetilde{M} \in \overline{\mathcal{F}_{\mathrm{inj}}}$.By $\otimes$-trick,
$\forall v_0 \in \mathcal{U}(H \otimes K, H)$, $\exists u_n \in \mathcal{U}(H \otimes K)$ s.t. $v_0 u_n^* \widetilde{M} u_n v_0^* \to M$ in vN(H). So $M \in \overline{\mathcal{F}_{\mathrm{inj}}}$.

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$\rightsquigarrow \widetilde{M} \overset{\leftarrow\ \varepsilon'}{\hookrightarrow_{i'}} Q^\omega$, $Q := R_\infty \overline{\otimes} \mathbb{B}(K_1) \overline{\otimes} \mathbb{C}1_{K_2}$. Since $Q, \widetilde{M}$: standard, $\exists w_n$ s.t. $w_n^* Q w_n \to \widetilde{M}$, so $\widetilde{M} \in \overline{\mathcal{F}_{\mathrm{inj}}}$. By $\otimes$-trick, $\forall v_0 \in \mathcal{U}(H \otimes K, H)$, $\exists u_n \in \mathcal{U}(H \otimes K)$ s.t. $v_0 u_n^* \widetilde{M} u_n v_0^* \to M$ in vN(H). So $M \in \overline{\mathcal{F}_{\mathrm{inj}}}$.

(3)$\Rightarrow$(2) As $\prod^\omega R_\infty$ has QWEP, $R_\infty^\omega \cong p(\prod^\omega R_\infty)p$ has QWEP too. So by $M \overset{\leftarrow\ \varepsilon}{\hookrightarrow_i} R_\infty^\omega$, M has QWEP.

Sketch: (1) $M \in \overline{\mathcal{F}_{\mathrm{inj}}} \Leftrightarrow$(3)$M \overset{\leftarrow\,\varepsilon}{\hookrightarrow}_i R_\infty^\omega \Rightarrow$(2) M: QWEP$\Rightarrow$(1).

(1)$\Rightarrow$(3) By HW '00, $\mathcal{F}_{\mathrm{III}_1} \cap \mathcal{F}_{\mathrm{inj}}$ is dense in $\mathcal{F}_{\mathrm{inj}}$, so $\exists M_n \cong R_\infty$ s.t. $M_n \to M$. Then $\exists \psi_n \in S_{\mathrm{nf}}(R_\infty)$ s.t. $N \overset{\leftarrow\,\varepsilon}{\hookrightarrow}_i (R_\infty, \psi_n)^\omega$, but $(R_\infty, \psi_n)^\omega \cong R_\infty^\omega$ by AH'12.

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(2)$\Rightarrow$(1) Assume M has QWEP. Use Haagerup-Junge-Xu '08 reduction method to reduce the problem to tracial case, and then use Kirchberg '93+our approximation thm! (Continued) □

Sketch of (2): M QWEP $\Rightarrow$(1) $M \in \overline{\mathcal{F}_{\mathrm{inj}}}$.

Assume M: QWEP. Use HJX'08 method: let $G = \mathbb{Z}[\frac{1}{2}] \subset \mathbb{R}$, and $\varphi \in S_{\mathrm{nf}}(M)$. Let $\widehat{\varphi}$ be the dual state on $N := M \rtimes_{\sigma^{\varphi}} G$. Then $M \overset{\overset{\varepsilon}{\leftarrow}}{\hookrightarrow}_{\pi} N$.

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By Kirchberg'93, $N_n \overset{\overset{\varepsilon_n}{\leftarrow}}{\underset{i_n}{\hookrightarrow}} R^{\omega}$, whence by AHW '13, $N_n \in \overline{\mathcal{F}_{\mathrm{inj}}}$.
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Then $M \overset{\leftarrow \varepsilon_0}{\hookrightarrow}_{i_0} R_{\infty}^{\omega}$ so $M \in \overline{\mathcal{F}_{\mathbf{inj}}}$ by (3)$\Rightarrow$(1). □

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By $\mathrm{vN}(\boldsymbol{H})^{\mathbf{QWEP}} = \overline{\mathcal{F}_{\mathbf{inj}}}$, $\mathrm{vN}(\boldsymbol{H})^{\neg\mathbf{QWEP}}$ is open. Assume $\exists M$ non-QWEP and $N \in \mathrm{vN}(\boldsymbol{H})$.

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Thank you for your attention!