Ultraproducts, QWEP von Neumann Algebras, and Effros-Maréchal Topology

Hiroshi ANDO

Erwin Schrödinger Institute, Vienna

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Joint work with Uffe Haagerup and Carl Winsløw
(University of Copenhagen)

Outline of Talk

- H. Ando, U. Haagerup, "Ultraprocucts of von Neumann algebras", arXiv:1212.5457
- H. Ando, U. Haagerup, C. Winsløw, "Ultraproducts, QWEP von Neumann algebras, and the Effros-Maréchal topology", arXiv:1306.0460

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The Polish space $\mathbf{vN}(\boldsymbol{H})$ is important in connecting descriptive set theory and the theory of von Neumann algebras (e.g. Sasyk-Törnquist '09 showed that the isomorphism relation on the Borel subset $\mathcal{F}_{\mathbf{II_1}} \subset \mathbf{vN}(\boldsymbol{H})$ of II_1 factors is complete analytic).

Quick Introduction to von Neumann algebras

$H \cong \ell^2$. $\mathbb{B}(H)$=algebra of all bounded linear operators $H \to H$, has *-algebra structure: $x + y, \lambda x, xy, x^*$.

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For a subset $\mathcal{S} = \mathcal{S}^* \subset \mathbb{B}(H)$, $\mathcal{S}''$ is the vNa generated by $\mathcal{S}$.

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- Type I_n $n = 1, 2, \cdots, \infty$ $M \cong M_n(\mathbb{C}),\ \mathbb{B}(H)\ (n = \infty)$
- Type II_1
  $\exists!\ \tau \colon M \to \mathbb{C}$ tracial state $(\tau(xy) = \tau(yx))$, $\dim(M) = \infty$.
- Type II_∞ $M \cong N \overline{\otimes} \mathbb{B}(\ell^2)$, N: type II_1.
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- Type III No nonzero tracial state.
- Type III factors are further classified into continuous family of Type III_λ $(0 \leq \lambda \leq 1)$ by Connes' S-invariant.

Typical Examples of non-type I factors:

Example (Group von Neumann algebras)

For Γ countable discrete group, consider the left regular representation $\lambda\colon \Gamma \to \mathbb{B}(\ell^2\Gamma)$ by

$$\lambda_g \delta_h := \delta_{gh}, \qquad g, h \in \Gamma.$$

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$L(\Gamma)$ is a II_1 factor$\Leftrightarrow$ Γ is ICC: $\sharp\{g^{-1}hg; g \in \Gamma\} = \infty$ for all $h \neq e \in \Gamma$ (e.g. $\mathfrak{S}_\infty = \varinjlim \mathfrak{S}_n, \mathbb{F}_n,$ $\mathbf{PSL}(n, \mathbb{Z})$ $(n \geq 2)$).

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$(1 \in) A$ C*-alg, a linear $\varphi\colon A \to \mathbb{C}$ is called a state if

- $\varphi(a^*a) \geq 0$ $(a \in A)$.
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- $\pi_\varphi\colon A \to \mathbb{B}(H_\varphi)$ is a *-representation.
- $\xi_\varphi \in H_\varphi$, $\|\xi_\varphi\| = 1$.
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Different φ defines different vNa $\pi_\varphi(A)''$.

Example (Type III factors)

Consider the inclusion

$$M_2(\mathbb{C}) \hookrightarrow M_2(\mathbb{C})^{\otimes 2} \hookrightarrow \cdots M_2(\mathbb{C})^{\otimes n} \hookrightarrow \cdots$$

Let $A = M_{2^\infty} = \overline{\bigcup_{n=1}^{\infty}}^{\|\cdot\|} M_2(\mathbb{C})^{\otimes n}$ be the CAR algebra.

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$$\varphi_\lambda|_{M_2(\mathbb{C})^{\otimes n}} = \mathrm{Tr}(\rho_\lambda \cdot)^{\otimes n}, \qquad n \geq 1,$$

where $\rho_\lambda = \frac{1}{1+\lambda}\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$.

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Then $M := \pi_{\varphi_\lambda}(A)''$ is

- Type II_1 factor if $\lambda = 1$. We denote it as $R = \bigotimes_{n=1}^{\infty}(M_2(\mathbb{C}), \mathrm{tr}_2)$ (hyperfinite II_1 factor).
- Type III_λ factor if $0 < \lambda < 1$. We denote it as $R_\lambda = \bigotimes_{n=1}^{\infty}(M_2(\mathbb{C}), \mathrm{Tr}(\rho_\lambda \cdot))$ (Powers factors)

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- hyperfinite if $M = \overline{\bigcup_{n=1}^{\infty}}^{\text{sot}} M_n$ for some increasing sequence $M_1 \subset M_2 \subset \cdots M$ of finite-dim *-subalgebras.

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- injective if $\exists E \colon \mathbb{B}(H) \to M$, $\|E\| = 1$, $E|_M = \mathrm{id}_M$. Such E is called a conditional expectation.

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Theorem (Connes '76)

Injectivity is equivalent to the hyperfiniteness. There exists only one isomorphism class of injective factors of type II_1, II_∞, III_λ $(\lambda \neq 0, 1)$

- Injective III_1 factor is also unique (Connes+Haagerup).

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- The isomorphism classes of injective III_0 factors are in 1-1 correspondence to the conjugacy class of properly ergodic flows (Krieger). Their classification problem is not smooth (Woods).

Theorem (Sasyk-Törnquist '10)

Even the infinite tensor product (=Araki-Woods) type III_0 factors (i.e., $M = \pi_\psi(M_{2^\infty})''$ for some $\psi = \bigotimes \psi_n$) are not classifiable by countable structures.

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Definition

Let (M, τ) II_1 factor. Fix $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$. Then the ultraproduct M^ω is

$$M^\omega = \ell^\infty(\mathbb{N}, M)/\{(x_n)_n; \lim_\omega \tau(x_n^* x_n) = 0\}.$$

M^ω is a (huge) II_1 factor with unique trace

$$\tau^\omega((x_n)^\omega) := \lim_\omega \tau(x_n), \quad (x_n)^\omega \in M^\omega.$$

By $M \ni x \mapsto (x, x, \cdots)^\omega \in M^\omega$, we regard $M \subset M^\omega$.
$M' \cap M^\omega$ is called the central sequence algebra of M.

Theorem (McDuff)

For a II_1 factor M, TFAE

- $M_\omega := M' \cap M^\omega$ *is non-commutative.*
- M_ω *is of type* II_1.
- $M \cong M\overline{\otimes}R$ *(M is McDuff).*

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> *... We now construct an approximate imbedding of N into R ($N \hookrightarrow R^\omega$). Apparently, such an imbedding ought to exist for all II_1 factors because it does for the regular representation of free groups...*
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Does every separable II_1 factor M embeds into R^ω?

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(End of the introduction)

QWEP Conjecture

Kirchberg ('93) revealed remarkable connetions among

- Tensor products of C*-algebras
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In this talk, we discuss how QWEP property is connected to ultraproducts of von Neumann algebras using topological method.

QWEP Conjecture

Definition (Lance '73, Kirchberg '93)

(1) C*-alg A has the weak expectation property (WEP) if for any faithful representation $A \subset \mathbb{B}(H)$, there is a ucp map $\Phi\colon \mathbb{B}(H) \to A^{**}$ s.t. $\Phi|_A = \mathrm{id}_A$.

(2) C*-alg A has the quotient weak expectation property (QWEP) if it is the quotient of a C*-algebra with WEP.

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Theorem (Kirchberg '93)

TFAE.

(1) $C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$.

(2) *Every C^*-algebra has QWEP.*

(3) *(Connes's Embedding) Every separable type II_1 factor M admits an embedding into R^ω, where $R = \bigotimes_{n=1}^\infty (M_2(\mathbb{C}), \mathrm{tr}_2)$.*

Theorem (Kirchberg '93)

A separable II_1 *factor* M *embeds into* R^ω *if and only if* M *has QWEP.*

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Is QWEP conjecture true?

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Theorem (Junge-Pisier '95)

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- Effros ('65) introduced Effros Borel structure on vN(H).
- Maréchal ('73) introduced Polish topology on vN(H) that generates Effros Borel structure.
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Definition (Maréchal '73)

The Effros-Maréchal Topology on vN(H) is the weakest topology which makes all the maps of the form

$$\mathrm{vN}(H) \ni M \mapsto \|\varphi|_M\|, \quad \varphi \in \mathbb{B}(H)_*$$

continuous.

Alternative description

d: wot-metric on Ball($\mathbb{B}(H)$).

$X := (\text{Ball}(\mathbb{B}(H)), d)$ is compact Polish. $\mathcal{K}(X)$=space of all compact subsets of X. Let δ be the Hausdorff metric on $\mathcal{K}(X)$.

$$\delta(K_1, K_2) := \max\left(\max_{x\in K_1} d(x, K_2), \max_{y\in K_2} d(y, K_1)\right).$$

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Then the EM-topology is metrisable by

$$\delta_{\mathrm{EM}}(M, N) = \delta(\mathrm{Ball}(M), \mathrm{Ball}(N)), \quad M, N \in \mathrm{vN}(H).$$

And Ball($\mathrm{vN}(H)$) is G_δ in $\mathcal{K}(X)$, whence Polish (Maréchal).

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Theorem (Maréchal '73, Haagerup-Winsløw '98)

*There exists a sequence of *sot-continuous $a_n : \mathrm{vN}(H) \to \mathrm{Ball}(\mathbb{B}(H))$ such that $\{a_n(M)\}_{n=1}^{\infty}$ is *-strongly dense in $\mathrm{Ball}(M)$.*

(cf. Kuratowski-Ryll-Nardzewski selection map)

Definition (Haagerup-Winsløw '98)

For $\{M_n\}_{n=1}^{\infty} \subset \text{vN}(H)$, define $\limsup\limits_{n\to\infty} M_n$ and $\liminf\limits_{n\to\infty} M_n$ by

(1) $\liminf\limits_{n\to\infty} M_n = \{x \in \mathbb{B}(H);\ x_n \xrightarrow{\text{so}*} x, \exists (x_n)_n \in \ell^{\infty}(\mathbb{N}, M_n)\}$.

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Theorem (Haagerup-Winsløw '98)

TFAE.

(1) $M_n \to M$ *in vN*(H).

(2) $\liminf_{n\to\infty} M_n = M = \limsup_{n\to\infty} M_n$.

Moreover, $\left(\limsup_{n\to\infty} M_n\right)' = \liminf_{n\to\infty} M_n'$ *holds.*

Theorem (Haagerup-Winsløw '98)

- $\mathrm{vN}(H) \times \mathrm{vN}(H) \ni (M, N) \mapsto M \overline{\otimes} N \in \mathrm{vN}(H^{\otimes 2})$ *is continuous.*
- $\mathrm{vN}(H) \ni M \mapsto M' \in \mathrm{vN}(H)$ *is a homeomorphism.*
- $\mathrm{vN}(H) \ni M \mapsto M \rtimes_{\sigma^{\varphi|_M}} \mathbb{R} \in \mathrm{vN}(L^2(\mathbb{R}, H))$ *is continuous.*

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Haagerup-Winsløw also showed that $\exists M_n\cong M_{k_n}(\mathbb{C})\in\mathrm{vN}(\ell^2\mathbb{F}_2)$ s.t.

$$\lim_{n\to\infty}M_n=L(\mathbb{F}_2)\quad \text{in } \mathrm{vN}(\ell^2\mathbb{F}_2)\quad (\heartsuit).$$

So even if M,N are δ_{EM}-close, they are in general very far from isomorphic.

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So even if M, N are δ_{EM}-close, they are in general very far from isomorphic.
In particular, the set $\mathcal{F}_{\mathrm{inj}}$ of injective factors is not a closed subset.
What is the meaning of $(\heartsuit)$? Is $\mathcal{F}_{\mathrm{inj}}$ Borel?

Important subsets of vN$(\boldsymbol{H})$: $\mathcal{F}$: factors, $\mathcal{F}_{\mathbf{inj}}$ injective factors, vN$(\boldsymbol{H})^{\mathbf{st}}$ standardly acting vNas, $\mathcal{F}_{\mathbf{II_1}}$ type $\mathsf{II}_{\mathbf{1}}$ factors, etc.

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Theorem (Haagerup-Winsløw '00)

Subset of vN$(\boldsymbol{H})$	*Dense in vN$(\boldsymbol{H})$?*	*$\boldsymbol{G_\delta}$?*
$\mathcal{F}$	*Yes*	*Yes*
$\bigcup_{n\le n_0}\mathcal{F}_{\mathbf{I}_n},\ n_0\in\mathbb{N}$	*No*	*Yes (closed)*
$\mathcal{F}_{\mathbf{I_{fin}}}$	*	*No (but $\boldsymbol{F_\sigma}$)*
$\mathcal{F}_{\mathbf{I}_\infty}$	*	*No (but $\boldsymbol{F_\sigma}$)*
$\mathcal{F}_{\mathbf{II_1}}$	*Yes*	*No*
$\mathcal{F}_{\mathbf{II}_\infty}$	*Yes*	*No*
$\mathcal{F}_{\mathbf{III_0}}$	*Yes*	*No*
$\mathcal{F}_{\mathbf{III}_\lambda},\ \lambda\in(0,1)$	*Yes*	*No*
$\mathcal{F}_{\mathbf{III_1}}$	*Yes*	*Yes*
$\mathcal{F}_{\mathbf{inj}}$	*	*Yes*
$\mathcal{F}^{\mathbf{st}}$	*Yes*	*Yes*

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$\mathcal{F}_{\mathbf{II_\infty}}$	*Yes*	*No*
$\mathcal{F}_{\mathbf{III_0}}$	*Yes*	*No*
$\mathcal{F}_{\mathbf{III_\lambda}},\ \lambda\in(0,1)$	*Yes*	*No*
$\mathcal{F}_{\mathbf{III_1}}$	*Yes*	*Yes*
$\mathcal{F}_{\mathbf{inj}}$	*	*Yes*
$\mathcal{F}^{\mathrm{st}}$	*Yes*	*Yes*

*Moreover, * are all equivalent to QWEP (Connes Embedding) conjecture.*

Theorem (Haagerup-Winsløw '00)

Let M be a II_1 factor on H. TFAE.

(1) $M \in \overline{\mathcal{F}_{\mathbf{inj}}}$.

(2) *There is an embedding* $i : M \to R^{\omega}$.

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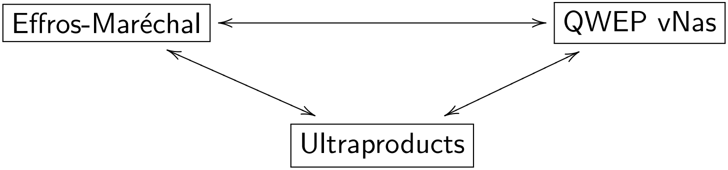
Goal of today's talk: further investigation of $\overline{\mathcal{F}_{\mathbf{inj}}}$ and study the connection of the following:



The Ocneanu Ultaproduct

$(M_n, \varphi_n)_{n=1}^{\infty}$ sequence of vNas/n.f.states. $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$.

Definition (Ocneanu '85)

Define Ocneanu ultraproduct
$(M_n, \varphi_n)^{\omega} := \mathcal{M}^{\omega}(M_n, \varphi_n)/\mathcal{I}_{\omega}(M_n, \varphi_n)$, where

$$\mathcal{I}_{\omega}(M_n, \varphi_n) := \{(x_n)_n \in \ell^{\infty}(\mathbb{N}, M_n);\ \|x_n\|_{\varphi_n}^{\sharp} \stackrel{n\to\omega}{\to} 0\},$$
$$\mathcal{M}^{\omega}(M_n, \varphi_n) := \{x \in \ell^{\infty}(\mathbb{N}, M_n);\ x\mathcal{I}_{\omega} + \mathcal{I}_{\omega}x \subset \mathcal{I}_{\omega}\}.$$

$\varphi^{\omega} \colon (x_n)^{\omega} \mapsto \lim_{\omega} \varphi_n(x_n)$ is a n.f. state on $(M_n, \varphi_n)^{\omega}$.

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Remark

Ocneanu considered the case $M_n \equiv M, \varphi_n \equiv \varphi$. In this case $(M, \varphi)^{\omega}$ is independent of φ, so denote it as M^{ω}.
However, it is crucial to use the sequence $\{\varphi_n\}_{n=1}^{\infty}$ to study Effros-Maréchal topology.

Ocneanu vs Groh-Raynaud

Given $\{M_n\}_{n=1}^{\infty}$, there is another construction $\prod^{\omega} M_n$, (Groh-Raynaud utlraproduct), s.t. $(\prod^{\omega} M_n)_* \cong$ Banach UP of $\{(M_n)_*\}_{n=1}^{\infty}$.

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For φ_n f.n. on M_n, let $\varphi_{\omega} = (\varphi_n)_{\omega} \in ((M_n)_*)_{\omega} = (\prod^{\omega} M_n)_*$.
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$p(\prod^{\omega} M_n)p \cong (M_n, \varphi_n)^{\omega}$ *holds.*

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(2) *If M is a type III$_0$ factor, then M^{ω} is never a factor.*

Ocneanu vs Groh-Raynaud

Given $\{M_n\}_{n=1}^{\infty}$, there is another construction $\prod^{\omega} M_n$, (Groh-Raynaud utlraproduct), s.t. $(\prod^{\omega} M_n)_* \cong$ Banach UP of $\{(M_n)_*\}_{n=1}^{\infty}$.
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4. *If M is a type III$_0$ factor, then $\exists \varphi_n$ s.t. $(M, \varphi_n)^{\omega}$ is tracial.*

Approximation Theorem

Connes's Embedding for general von Neumann algebras?

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Definition

For vNas M, N, we write $N \overset{\leftarrow \varepsilon}{\hookrightarrow}_i M$ if there is an embedding $i : N \to M$ and a normal faithful conditional expectation $\varepsilon : M \to i(N)$.

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Remark

- *If $M \overset{\leftarrow \varepsilon}{\hookrightarrow}_i \mathbb{B}(\ell^2)$, then M is atomic.*
- *If $N \overset{\leftarrow \varepsilon}{\hookrightarrow}_i M$ and M is semifinite, then N is also semifinite (Sakai, Tomiyama).*

Proposition (A-Haagerup-Winsløw '13)

Suppose $M_n \to N$ *in* $\mathrm{vN}(H)$*. Then for any n.f. state* $\chi \in \mathbb{B}(H)_*$*,*

$$N \overset{\leftarrow\ \varepsilon}{\hookrightarrow}_i (M_n, \psi_n)^\omega,$$

where $\psi_n := \chi|_{M_n}$ *and moreover* $\varphi = (\psi_n)^\omega \circ i$*,* $\varphi := \chi|_N$ *holds.*

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Sketch.

(Construction of i) Let $x \in N = \lim M_n$. Then $N = \liminf M_n$. So $\exists (x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$ s.t. $x_n \overset{\text{so*}}{\to} x$. Then $(x_n)_n \in \mathcal{M}^\omega(M_n, \psi_n)$ and $i\colon N \ni x \mapsto (x_n)^\omega \in (M_n, \psi_n)^\omega$ is well-defined.

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(Construction of ε) Given $x = (x_n)^\omega \in (M_n, \psi_n)^\omega$, let $\tilde{x} :=$ wo- $\lim_\omega x_n$. Then as $N = \limsup M_n$, $\tilde{x} \in N$ and $\varepsilon\colon (M_n, \psi_n)^\omega \ni x \mapsto i(\tilde{x}) \in i(N)$ is well-defined.
By direct calculations, we have $N \overset{\overleftarrow{\varepsilon}}{\hookrightarrow}_i (M_n, \psi_n)^\omega$. □

Corollary (A-Haagerup-Winsløw '13)

Let $M, N \in \mathrm{vN}(H)$ *and assume* $M_n \cong M$ $(\forall n)$ *and* $M_n \overset{n\to\infty}{\to} N$ *in* $\mathrm{vN}(H)$. *Then* $\exists\{\psi_n\}_{n=1}^{\infty}$ *s.t.*

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As a (partial) converse:

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As a (partial) converse:

Proposition (A-Haagerup-Winsløw '13)

Suppose $M_n,\ N \in \mathrm{vN}(H)^{\mathrm{st}}$ *and* $\exists \psi_n \in S_{\mathrm{nf}}(M_n)$ *s.t.* $N \overset{\leftarrow\ \varepsilon}{\hookrightarrow}_i (M_n, \psi_n)^{\omega}$. *Then* $\exists u_n \in \mathcal{U}(H)$ *and* $\exists n_1 < n_2 < \cdots$ *s.t.*

$$u_{n_k} M_{n_k} u_{n_k}^* \overset{k\to\infty}{\to} N \quad \text{in} \ \ \mathrm{vN}(H).$$

With additional efforts, we get the main result:

Theorem (A-Haagerup-Winsløw '13)

For $M \in \mathrm{vN}(H)$, *and* $0 < \lambda < 1$. *TFAE.*

(1) $M \in \overline{\mathcal{F}_{\mathbf{inj}}}$.

(2) M *has QWEP.*

(3) $M \overset{\leftarrow\,\varepsilon}{\hookrightarrow_i} R_\infty^\omega$. R_∞: *hyperfinite* III_1 *factor.*

(4) $M \overset{\leftarrow\,\varepsilon}{\hookrightarrow_i} R_\lambda^\omega$. R_λ: *hyperfinite* III_λ *factor.*

(5) $M \overset{\leftarrow\,\varepsilon}{\hookrightarrow_i} (M_{k_n}(\mathbb{C}), \varphi_n)^\omega$ *for some* $\{k_n\}_{n=1}^\infty$ *and* $\varphi_n \in S_{\mathrm{nf}}(M_{k_n}(\mathbb{C}))$.

(6) $\forall \varepsilon > 0$, $\forall n \in \mathbb{N}$, $\forall \xi_1, \cdots, \xi_n \in \mathcal{P}_M^\natural$, $\exists k \in \mathbb{N}$ *and* $\exists a_1, \cdots, a_n \in M_k(\mathbb{C})_+$ *s.t.*

$$|\langle \xi_i, \xi_j \rangle - \mathrm{tr}_k(a_i a_j)| < \varepsilon \qquad (1 \leq i, j \leq n).$$

Here, $\mathcal{P}_M^\natural$ *is the natural cone in the standard form of* M.

Some consequences:

Corollary

If QWEP conjecture fails, then $\mathrm{vN}(H)^{\neg \mathbf{QWEP}} = \{M; M \neq \mathbf{QWEP}\}$ *is open dense in* $\mathrm{vN}(H)$.

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Farah-Hart-Sherman proved by model theory, that:

Theorem (Farah-Hart-Sherman '11)

$\exists M \in \mathcal{F}_{\mathbf{II_1}}$ *s.t.* $N \hookrightarrow M^{\omega}$ *for every* $N \in \mathcal{F}_{\mathbf{II_1}}$.

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We prove by EM-topology instead, that:

Theorem (A-Haagerup-Winsløw'13)

$\exists M \in \mathcal{F}_{\mathrm{III}_1}$ *s.t.* $N \overset{\leftarrow\varepsilon}{\hookrightarrow}_i M^{\omega}$ *for every* $N \in \mathrm{vN}(H)$.

Proof.

Choose (by HW '00) a sequence $\{M_n\}_{n=1}^{\infty}$ of III_1 factors which is dense in $\mathrm{vN}(H)$ and $\varphi_n \in S_{\mathrm{nf}}(M_n)$, and put $(M, \varphi) := \bigotimes_{\mathbb{N}}(M_n, \varphi_n)$.

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Then $\forall N$, $\exists\{n_k\}_{k=1}^{\infty}$ s.t. $M_{n_k} \to N$, and $(M_{n_k},\varphi_{n_k}) \overset{\leftarrow\,\varepsilon_k}{\underset{i_k}{\hookrightarrow}} (M,\varphi)$.

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Then by approximation thm, one has $\exists \psi_{n_k} \in S_{\mathrm{nf}}(M_{n_k})$ s.t.

$$N \overset{\overleftarrow{\varepsilon_0}}{\hookrightarrow}_{i_0} (M_{n_k},\psi_{n_k})^{\omega} \overset{(\heartsuit)}{\cong} (M_{n_k},\varphi_{n_k})^{\omega} \overset{\overleftarrow{(\varepsilon_k)^{\omega}}}{\hookrightarrow}_{(i_k)^{\omega}} (M,\varphi)^{\omega} \cong M^{\omega}.$$

Here, $(\heartsuit)$ is by Connes-Størmer. □

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There are many Polish (or Borel) subspaces of $\mathrm{vN}(H)$ that have not been well studied.